PREFACE

These notes are the English version of notes for a one-semester undergraduate course through the medium of Irish which is to be offered for the first time at NUI Maynooth in the 2006–07 academic year. The writing of both sets of notes was made possible by the kind support of Bord Gaeilge na hOllscoile, NUI Maynooth.

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Chapter 1

Introduction

1.1 Fractals

A **fractal** is a set that remains "complicated" at all scales no matter how small. One example of a fractal is any natural coastline. If you look at a map of the world, you see that the outline of the continents is complicated in the sense that it is not well approximated by a few line segments. A map of Europe is equally complicated: compared with a map of the whole world, it has lost all the complications of the other continents, but it has gained complications that were too small to show up in the map of the whole world. In a similar fashion a map of Ireland or just a part of the coastline of Ireland gain smaller scale complexity to compensate for the loss of larger scale complexity.

The fact that small scale maps and large scale maps of coastlines both look about equally complicated is mirrored in many other areas of nature. For instance under a powerful microscope a single cell from a tree or a single elephant hair look as complicated as the whole tree or elephant do to the naked eye. Such complexity across all (or more truthfully, a large number of) scales is the most significant difference between most real-life objects and most mathematical sets and functions that you have seen in all your previous mathematics courses. For instance, suppose you plot in Maple the graph of pretty much any function $y = f(x)$ that you can think of. Whether you use a polynomial, a trigonometric function, an exponential or log, or some complicated sum of such functions, you'll notice that whatever complications that are in the graph when you plot it over a large interval disappear if you pick a sufficiently small interval. For instance a command of the form

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plot(f(x), x=-10..10)
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might give a complicated looking plot (depending on the choice of f), but for most functions f that you are likely to choose,

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plot(f(x), x=-2.376..-2.375)
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will produce something very like a straight line. As another example, consider the set of points (x, y, z) satisfying $x^2 + y^2 + z^2 = r^2$ for some $r > 0$. This is a sphere of radius r about the origin, and so very curved, but if we look at any very small part of it, it looks flat like a piece of a plane (hence the belief for a long time that the Earth is flat). Sets that look flatter and flatter at smaller and smaller scales are said to be **smooth** and can be viewed as the exact opposite of fractals.

For most of its history, mathematics steered clear of fractals and mainly concentrated on sets that possessed some degree of smoothness. Indeed the French term *fractal/fractale* (adjective/noun) was only invented in 1975 by Benoît Mandelbrot (1924–), the Polish-French mathematician. The advent of modern computers and graphical displays have greatly helped and inspired the study of fractals.

Nevertheless there are some much earlier examples of fractal sets. In 1872, the German mathematician Karl Weierstrass (1815–1897) defined a nowhere differentiable function that was everywhere continuous; the graph of that function would now be called a fractal. Weierstrass' example was hard to visualize but a much simpler example was found by the Swedish mathematician Helge von Koch (1870–1924) in 1904. The idea of an iterated function system is an important way of defining fractals and it was investigated by people such as Poincaré, Fatou, and Julia in the late 19th and early 20th centuries, and we will discuss it in more detail in Chapters 2 agus 3.

Such sets were viewed as isolated bizarre sets until Mandelbrot highlighted their common features: for instance, most are self-similar and their dimension is not a whole number. We will discuss in Chapter 3 what a fractional dimension can mean but such notions go back to the early part of the 20th century. Mandelbrot also popularized fractals by showing how they were realistic and useful models of many natural phenomena, including

- the shape of coastlines and other geological features;
- the structure of plants, blood vessels, and lungs;
- stock market prices;
- Brownian motion.

He argued that fractals were more intuitive and natural than the smooth objects of traditional mathematics. On page 1 of *The Fractal Geometry of Nature* [11], Mandelbrot says:

> Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

Since fractals are inherently complicated, it is good to have some fairly simple concepts that summarize some of their more important features. One of the most important such concepts is the dimension of a set. There are actually several concepts of dimension but all agree with the intuitive concept of dimension for smooth sets. For instance, the dimension of the graph of a smooth function $y = f(x)$ is 1 (because a small part of it always looks like a line segment), and the dimension of a sphere is 2 (because a small part of it looks planar). However the dimension of a fractal is usually not an integer! We will discuss a couple of notions of dimension in Chapter 3. The notion of Minkowski dimension is simpler than that of Hausdorff dimension, but the latter has nicer properties and also allows us to generalize concepts such as length and area to fractals.

1.2 Dynamical Systems and Chaos

You have probably heard of the **Butterfly Effect**, now a popular and vivid synonym for chaos. Indeed *The Butterfly Effect* was the title of a 2004 movie starring Ashton Kutcher. The idea itself is usually ascribed to the American meteorologist Edward Lorenz (1917–) who gave a talk in 1972 entitled *Predictability: Does the Flap of a Butterfly's Wings in Brazil Set off a Tornado in Texas?* Using a butterfly to indicate this idea may have been inspired by the 1952 Ray Bradbury short story *A Sound of Thunder* in which a time traveller accidentally steps on a butterfly in the distant past, causing big changes to the present. Lorenz' idea was based on the fact that weather seems to be one of those initial value systems where two sets of initial conditions A and B that differ very little (perhaps by as little as the flap of a butterfly's wings) can cause the associated solutions (i.e. future weather conditions) to evolve in such a way that this little difference grows over time until eventually it is very large (perhaps differing by a tornado) after a sufficiently long period of time (such as a year).

Mathematical modelling of the weather was in its infancy in the 1960s when Edward Lorenz ran simple experimental models on his computer. After running one particular sequence, he decided as a good scientist to replicate it, so he re-entered a number from his printout, taken half-way through a sequence of time intervals, and left it to run. Computers ran more slowly back then so he went away and was surprised when he returned to find results that were very different from his first experiment. Eventually he realized the reason: in a moment of laziness, he had entered the number 0.506 instead of the printout figure 0.506127. He had assumed that the small error would have stayed small in the future but instead it had got much larger. After more experimentation, Lorenz concluded that the slightest difference in initial conditions—even in, say, the 20th decimal place and so much to small to measure—would eventually lead to completely different outcomes for the solution. Since one can only hope to minimize, but never eliminate, measurement error, it follows that although this system was predictable in the short term (i.e. one could solve the equations at least numerically to any desired degree of accuracy), long-term there was no hope for a solution. This violated the basic beliefs of physics and so was rather revolutionary. There are many such systems with such sensitive dependence on initial conditions and this is one of the defining features of so-called chaotic systems as opposed to stable systems.

Discussions of sensitive dependence on initial conditions and stability of systems go back much further. Oscar II, King of Sweden and Norway, held a mathematical competition in 1889 to celebrate his sixtieth birthday. The prize was to be given to the best contribution related to any of a number of hard mathematical problems. The famous French mathematician Henri Poincaré (1854–1912) was awarded the prize for a paper on the 3-body problem (describing the motion of three bodies, such as the Sun and two planets, under gravitation). Here he developed significant parts of what we now call the theory of dynamical systems and chaos theory, and in particular showed that the motion of three bodies could be unstable and chaotic.

The background to Poincaré's work begins with the famous English mathematician/physicist Isaac Newton (1643–1727), who proved that a solar system of two bodies (i.e. the sun and one planet) is periodic (and so perfectly stable). The problem of three or more bodies is much more difficult and we now know that it cannot be solved exactly. Despite the impossibility of finding exact solutions, one could still hope to answer the question of whether this system is stable or unstable. The system would be unstable if for instance it exhibited sensitive dependence on initial conditions, or even worse: will the Earth always remain about the same distance from the Sun, or will the gravitational effects of the larger planets eventually cause us to spiral into the sun or out to the cold depths of space? The great French mathematician Pierre-Simon Laplace (1749–1827) tried to solve this problem by making some simplifying assumptions but unfortunately these very assumptions invalidated his work. It was not after Poincaré's work that it began to become clear that just about any reasonable looking simplification of the problem would make the solution completely invalid in the long term. Essentially there are no short-cuts to the calculation of the long-term evolution of a chaotic system from given initial conditions.

There has been some more recent progress on the problem of the stability of the solar system. The mathematicians Andrei Kolmogorov (Russian; 1903–1987), Vladimir Arnol'd (Russian; Kolmogorov's student; 1937–) and Jürgen Moser (German-American; 1928–1999) are responsible for what we now call **KAM theory**. This powerful theory in particular says that sufficiently tame solar systems are stable, but it does not give an answer for our own solar system. On the other hand in wilder solar systems there are certainly problems: if Jupiter were 100 times more massive, the Earth's orbit would change erratically to the extent that life as we know it would be impossible. Is our solar system on the right side of the dividing line between tame and wild systems? We do not know! Of course even if the answer were that it is tame, over very long periods of time we also have to worry about gravitational effects external to our solar system: a star might wander too close and upsets our solar system, or a nearby dwarf galaxy might cause great upheaval throughout our part of the Galaxy.

Returning to the relative simplicity of our solar system in isolation, the French mathematician Jacques Lasker estimated numerically in 1989 that an initial error of 15 meters in the orbital position of the Earth would make it impossible to predict where the Earth would be in its orbit in just over 100 million years' time.

A lot of systems can be viewed abstractly at a particular point in time as a set of data D (for instance in the case of the Earth's orbit the data would give velocity and position relative to the sun), together with a function f that gives the new data one unit of time into future (perhaps a year in the case of the Earth). If we want to go two units of time into the future, the new data is then $f(f(D))$ which we write as $f^2(D)$ for brevity. The question of where the system is 100 million units of time into the future amounts to calculating the value of $f^{100\,000\,000}(D)$. We are usually not so much interested in calculating this as in knowing whether $f^{100\,000\,000}(D')$ must be close to $f^{100\,000\,000}(D)$ if D' and D are two sets of data that are close to each other. For stable systems, the answer is yes, but for chaotic systems, the answer is typically no.

Lorenz advocated the examination of simple quadratic maps such as the logistic map $g : \mathbf{R} \to \mathbf{R}$, $g(x) = \lambda x(1-x)$ for some $\lambda > 0$, as good models for more complicated chaotic systems such as the weather. Today we understand the dynamics of the logistic map quite well, and we will discuss this topic at some length in these notes.

1.3 These notes

Almost all the proofs in these notes follow from the basic theory of continuous and differentiable functions on **R** seen in first year calculus. We do occasionally mention metric spaces but most of the time we use only the usual Euclidean metric (distance) on **R**. We carry out some explorations in Maple, so you will need access to Maple to fully appreciate and explore these parts.