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Preface

This book is really a second, revised edition of the Maynooth Mathematical Olympiad Manual. The original edition proved generally useful, and sold over 1000 copies. Most sales were in Ireland, but there has also been interest from over 20 other countries. The main difference in the new edition is the replacement of Chapter 11, on Functional Equations, by an expanded version. This includes contributions from James Cruickshank, of NUI Galway. Another author, Gary McGuire, has moved from Maynooth to UCD, so it seemed appropriate to change 'Maynooth' to 'Irish' in the title. Apart from that, minor improvements and corrections have been made to the rest of the book. The editor would like to acknowledge the helpful comments and corrections sent in by many readers of the original edition, both students and trainers. Particularly valuable were those sent by Finbarr Holland, of UCC.

Preface to the Maynooth Mathematical Olympiad Manual

This book is primarily intended to assist Irish secondary–school students who are preparing to compete in the Irish Mathematical Olympiad (held in May each year) or the International Mathematical Olympiad (held in July each year). It may also be of interest to others who enjoy mathematics.

The Olympiads are written examinations, based on 'second–level mathematics'. There are significant variations between countries in the content of second–level programmes in Mathematics. Thus, Irish competitors find themselves faced with problems that require background knowledge that is not covered in the Senior Cycle programme for our schools. In order to have a reasonable chance of success, they need to master this material.

There are many problem–collections available, which students may use to hone their problem–solving skills. Particularly useful is the collection *Irish Mathematical–Olympiad Problems 1988–1998*, edited by Finbarr Holland and published by the IMO Irish Participation Committee in 1999. There are also some good books which provide some background information in combination with problems and advice on problem–solving. In our training sessions at Maynooth, we have found Derek Holton's book *Let's Solve Some Math Problems* (University of Waterloo, 1993) useful for beginners. Unfortunately, this book is now out of print. For more advanced problem–solvers, we found *Mathematical Circles (Russian Experience)* by D. Fomin, S. Genkin and I. Itenberg (Amer. Math. Soc. 1996) particularly useful as a source of good problems, nicely graduated. However, there is no book specifically designed to bridge the gap faced by Irish students between the material normally covered in school and the material they need to know. That is what this book is intended to do.

The individual authors are responsible for the following chapters: David Wraith for Chapter 4 and Chapter 7; Richard Watson for Chapter 5; David Redmond for Chapter 6; Gary McGuire for Chapter 8; and Anthony G. O'Farrell for the rest, and for the editorial work.

Some material for this book is translated from the book, written in Irish: by Antóin Ó Fearghaíl (= AGO'F): *Nótaí an Bhráthar Mac Craith* (Forthcoming, Marino Institute of Education, 2002, available from the Mathematics Department, NUIM). For those who can read Irish, that book has additional material in geometry (such as the theory of pole and polar, and the theory of the parabola) which is useful for some problems that occur in Olympiads. It also covers some calculus, which is never essential for Olympiad problems, but may be used if the student happens to know it.