A CONTROLLABILITY CRITERION FOR SWITCHED LINEAR SYSTEMS

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ABSTRACT. We report sufficient conditions on a switching signal that guarantee that the solution of a switched linear system converges asymptotically to zero. These conditions apply to continuous, discrete-time and hybrid switched linear systems, with either entirely stable subsystems or a mixture of stable and unstable subsystems. The conditions are general enough to allow engineers to design switching signals that make switched systems controllable.

1. INTRODUCTION

In Science and Engineering one frequently meets systems that consist of a family of subsystems and a switching signal which determines which subsystem is activated at each time.

When all the subsystems are linear, one has a *switched linear system*

$$\dot{x}(t) = A_{\sigma(t)}x(t) \tag{1}$$

where $\sigma : [0, +\infty) \to \{1, \cdots, n\}$ is the switching signal and $A_i : \mathbb{R}^m \to \mathbb{R}^m \quad (i = 1, \cdots, n)$ are matrices that characterise the subsystems.

The large number of areas in which switched linear systems appear makes their study a matter of real concern and great importance [3, 8, 12]. Its theoretical importance [3, 9, 10, 11, 13] derives from its practical importance: one needs to understand under what circumstances the system (1) is stable, or what switching signals make the systems stable.

Liberzon and Morse [3] formulated three basic problems in relation to the stability of switched systems.

"Problem A: Find conditions that guarantee that the switched system is asymptotically stable for any switching signal".

"Problem B: Identify those classes of switching signals for which the switched system is asymptotically stable".

"Problem C: Construct a switching signal that makes the switched system asymptotically stable".

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The condition of asymptotic stability referred to in *Problem A*, is desirable in practical applications. Unfortunately, the theorems that provide solution (or partial solutions) to *Problem A* involve conditions that are either computationally-infeasible (such as the existence of general Lyapunov functions, or conditions on the joint spectral radius of the family of matrices [1, 2], or too restrictive for many applications (such as the existence of Lyapunov functions in particular forms, symmetric systems, pairwise commutativity of the subsystems, and Lie-algebraic conditions, [4, 5, 6, 7, 9, 10]). On the other hand, it is well-known that there exist systems that exhibit instability even though all their subsystems are asymptotically stable [3, 4]. As a result, one sees the necessity of solving *Problem B* in practice, in order to deal with the applications. More often than not, *Problem B* is studied under the assumption that all the individual subsystems are asymptotically stable [3, 6]. However, for some applications it is convenient to allow subsystems that may be stable or unstable.

In this paper, we establish conditions on the switching signal of a switched linear systems that are sufficient to ensure asymptotic stability. We allow both stable and unstable subsystems. Our analysis will apply both to random and to deterministic switching signals $\sigma(t)$.

The paper is organised as follows. First we work on continuous switched linear systems, then on discrete systems. Afterwards we combine these to study hybrid systems. Then we apply our results about problems of *type B* to the design of switching signals in order to solve problems of *type C*.

2. Continuous Systems

Consider a continuous-time system (1). We shall refer to "switching on and off" the i-th subsystem or the matrix A_i, in the obvious sense: the i-th subsystem is "on" whenever $\sigma(t) = i$, and switching occurs when the value of $\sigma(t)$ changes. It is immaterial for the evolution of the system which value is taken by $\sigma(t)$ at these switching times. We will also say that the system is "ruled by" the matrix A_i when the i-th subsystem is on.

We make some basic assumptions:

Assumption 1: We assume that there are a finite number of switches in each finite time interval

This rules out "dithering"behaviour at arbitrarily-short time-scales [3, 4] . (However, we will allow instantaneous changes, or shocks to the system when we consider hybrid systems later).

This assumption allows us to define functions n_i, as follows:

Definition 1. *$n_i(t)$, for $t \geq 0$, denotes the number of disjoint (completed or under way) time periods up to and including time t during which the matrix A_i is switched on.*

Assumption 2: We also assume that the system switches on each subsystem infinitely often.

This is the same as saying that each $n_i(t)$ tends to $+\infty$ as $t \uparrow \infty$.

This condition makes sense from a practical point of view, and is not a real restriction in practice. If some subsystem is not used after a given time, then it can be dropped from the analysis without affecting the outcome, as regards asymptotic stability.

For each i, the matrix A_i will be switched on repeatedly. We need notation for the lengths of time it is used.

Definition 2. *Let t_{ij} denote the duration of the j-th time period during which the system is ruled by matrix A_i.*

Thus t_{ij} $(j = 1, 2, 3, \ldots)$ is an infinite sequence of positive real numbers.

Definition 3. *We denote by $m_i(t)$ the total duration of the periods up to time t for which the i-th subsystem is switched on.*

Thus, if time t is the end of the j-th period during which A_i is switched on, $m_i(t)$ will equal the sum $t_{i1} + \cdots + t_{ij}$. Thereafter, $m_i(t)$ will remain constant until the beginning of the next period when A_i is switched on, and will then start increasing with derivative 1.

We denote by $\|x\|$ the norm of $x \in \mathbb{R}^m$, with respect to some fixed norm on $\mathbb{R}^m$, and by $\|A\|$ the induced norm of an m by m matrix A:

$$\|A\| = \sup\{\|Ax\| : \|x\| \leq 1\}.$$

For instance, if we use the usual Euclidean norm on $\mathbb{R}^m$, then $\|A\|$ is $\sqrt{\lambda}$, where λ is the largest eigenvalue of A^*A.

The norm $\|\cdot\|$ determines n one-parameter functions $t \mapsto \|e^{tA_i}\|$, which we refer to as the norms of the flows corresponding to the n subsystems. The switching function σ determines, for each time t, the time-weighted geometric mean of the norms of the flows in the i-th subsystem up to the last switch at or before that time, which we denote by

$$\langle e^{A_i} \rangle = \langle e^{A_i} \rangle(t) := \left[\prod_{i_j=1}^{n_i} \left\| e^{A_i t_{ij}} \right\| \right]^{\frac{1}{m_i}}$$

We let the asymptotic limit of these means be

$$c_i = \limsup_{t\to\infty} \langle e^{A_i} \rangle.$$

Remark 1. *One could include a factor in the definition of $\langle e^{A_i} \rangle$ to account for the change since the last switch, or use the value of m_i at the last switching-time. These variant definitions produce quantities which will not differ materially from one another under the conditions of the theorem stated below. The present version is easiest to use, in practice.*

We observe that each $c_i < +\infty$. In fact, c_i is bounded by a constant (depending only on the norm used) times the spectral radius of the matrix e^{A_i} (the maximum of the absolute values of its eigenvalues). For a similar reason, each c_i will be bounded away from zero.

It is of crucial importance for our stability analysis whether some of the c_i are less than 1. If A_i is a *Hurwitz* matrix, i.e. has all its eigenvalues in the left half-plane, then $\|\exp(tA_i)\| < 1$ when t is large enough, so one may arrange that $c_i < 1$ by insisting that all the t_{ij} stay greater than a suitable lower bound. However, it may well happen that $c_i > 1$ for a Hurwitz A_i, depending on the norm used and the t_{ij}.

Let $d_i = \max\{c_i, 1\}$. We will use d_i instead of c_i when we do not wish to rely on the stability of subsystem i to stabilize the entire system.

Now let

$$\mu_i = \liminf_{t\uparrow\infty} \frac{m_i(t)}{t}, \qquad \nu_i = \limsup_{t\uparrow\infty} \frac{m_i(t)}{t},$$

for $i = 1, \ldots, n$. These quantities are asymptotic bounds for the proportion of the time that the i-th system is switched on.

Assumption 3. We assume that for some integer k with $1 \le k \le n$, we have $c_i < 1$ for $i \le k$, and

$$c_1^{\mu_1} \cdots c_k^{\mu_k} \cdot d_{k+1}^{\nu_{k+1}} \cdots d_n^{\nu_n} < 1. \tag{2}$$

(Here, we employ the usual convention, according to which the empty product equals 1; thus if $k = n$, the product $d_{k+1}^{\nu_{k+1}} \cdots d_n^{\nu_n} = 1$.)

We call the systems $A_1,\ldots,A_k$ the stabilizing systems. The others we refer to as bad (even though they may be either unstable systems or systems with $c_i < 1$ that we don't use to stabilize the whole).

Assumption 3 says, *inter alia*:

- There is at least one stabilizing system.

- If there is any unstable system, then some stable system is switched on for at least a fixed proportion of any sufficiently-long time period.
- Over long time periods, the stabilizing systems do enough (in a rather crude sense) to dampen out the effects of the unstable systems.

For instance, the assumption will hold if $1 \leq k < p \leq n$, $c_i < 1$ when $i < p$, and

$$c_1 \cdots c_k \left(c_p \cdots c_n\right)^s < 1,$$

and for each $j \geq p$ and each $i \leq k$

$$\liminf_{t \uparrow \infty} \frac{m_i(t)}{m_j(t)} > \frac{1}{s}.$$

In other words, for each unstable system A_j and each stabilizing system A_i, the system A_j is used for less than s times as long as A_i over all long time periods. A simple example of this is when

$$c_1 \cdot \left(c_p \cdots c_n\right)^s < 1,$$

and the system A_1 has $\mu_1 > 1/s$, i.e. is used for more than a proportion $1/s$ of the time, in the large. This may be used, rather brutally, to stabilize a given system by adding a very stable matrix and insisting that it be used often enough.

More generally, given a system with $c_i < 1$ for $i \leq k$ and

$$c = c_1^{\mu_1} \cdots c_k^{\mu_k} \cdot d_{k+1}^{\nu_{k+1}} \cdots d_n^{\nu_n} > 1,$$

one may stabilize it by adding a single stable system A_0, choosing $t_0 > 0$ such that

$$\|e^{t_0 A_0}\| = \lambda < 1,$$

and switching on A_0 for t_0 time-units (seconds, microseconds, or whatever is appropriate to the application) in every period of $N t_0$ time-units, so that $c_0 = \lambda$; if the rest of the system is run as before in the remaining $(N-1)t_0$ time-units of each period, then the new switched system will be stable if

$$c_0^{\frac{1}{N}} \cdot c < 1.$$

Obviously, there will be a trade-off between the severity of the damping (greater if c_0 is less) and the proportion of time that must be devoted to damping.

The following theorem gives us sufficient conditions to control a continuous system.

Theorem 1. *Consider a switched linear system of the form (1) that satisfies Assumptions 1, 2 and 3. Then each solution $x(t)$ of (1) tends asymptotically to zero.*

Proof. After a time t, for each i, the transfer matrix A_i will have been used for $n_i(t)$ complete time periods. One of the matrices will be currently in use for the (n_i+1)-st time. Taking norms we obtain

$$\|x(t)\| \leq \prod_{j=1}^{n_1} \left\| e^{A_1 t_{1j}} \right\| \cdots \prod_{j=1}^{n_n} \left\| e^{A_n t_{nj}} \right\| \cdot \|x_0\| . \tag{3}$$

Let

$$c := c_1^{\mu_1} \cdots c_k^{\mu_k} \cdot c_{k+1}^{\nu_{k+1}} \cdots c_n^{\nu_n}.$$

By Assumption 3, we may choose κ with $c < \kappa < 1$.

Choose $\varepsilon_1 > 0$ such that for each $i \leq k$ we have $\varepsilon_1 < 1 - c_i$, and

$$(c_1+\varepsilon_1)^{\mu_1-\varepsilon_1} \cdots (c_k+\varepsilon_1)^{\mu_k-\varepsilon_1} \cdot (c_{k+1}+\varepsilon_1)^{\nu_{k+1}+\varepsilon_1} \cdots (c_n+\varepsilon_1)^{\nu_n+\varepsilon_1} < \kappa.$$

Fix $\varepsilon > 0$. Fix $x_0 \in \mathbb{R}$.

Choose $M > 0$ such that $\kappa^M \|x_0\| < \varepsilon$.

Choose $T > 0$ such that $t > T$ implies that for each $i \in 1, \ldots, n$, we have

$$\langle e^{A_i} \rangle \leq c_i + \varepsilon_1$$

and for $1 \leq i \leq k$, we have

$$\frac{m_i(t)}{t} > \mu_i - \varepsilon_1$$

and for $k < i \leq n$,

$$\frac{m_i(t)}{t} < \nu_i + \varepsilon_1.$$

Then for $t > T$, we have

$$\|x(t)\| \leq \prod_i \langle e^{A_i} \rangle^{m_i} \|x_0\|.$$

In view of the fact that $c_i + \varepsilon_1$ is less than 1 when $i \leq k$ and is greater than 1 when $i > k$, we can bound the product from above by

$$\left\{ \prod_{i=1}^{k} (c_i+\varepsilon_1)^{\mu_i-\varepsilon_1} \cdot \prod_{i=k+1}^{n} (c_i+\varepsilon_1)^{\nu_i+\varepsilon_1} \right\}^t < \kappa^t.$$

Thus if $t > \max\{M, T\}$, we have $\|x(t)\| < \epsilon$. Hence $x(t) \to 0$ as $t \uparrow \infty$, as required. □

3. Discrete-time systems

When time is discrete instead of continuous we have a switched linear discrete-time system, in which the system (1) is replaced by

$$x(n+1) = A_{\sigma(n)}x(n) \tag{4}$$

where $\sigma(n)$ is now a switching signal defined for positive integral times, and A_i are $m \times m$ matrices, as before.

Discrete-time systems are as useful in engineering as continuous-time systems, and theoretical research is also very active. Furthermore, they appear in other areas where continuous system are not found, for example as a result of using the transfer matrix method to solve differential equations [14]. Lately, they are becoming more important in the study of structures consisting of stiffened plates (naval architecture, bridge engineering, aircraft design) [15] and spatially periodic structures (satellite antennae, satellite solar panels) [16]. The theorem, stated below, will indicate to designer how to insert panels (given by A_i in (4)) so that oscillations fade off and do not damage the structure.

The notation and assumptions of the last section can be adapted for discrete systems, as follows.

There is no need for Assumption 1.

Definition 4. *For an integral time t, $n_i(t)$ denotes the number of $j \leq t$ for which $\sigma(j) = i$.*

Assumption 2': We assume that the system uses each subsystem infinitely often, i.e that each $n_i(t) \uparrow \infty$.

Now let

$$\mu_i = \liminf_{t\uparrow\infty} \frac{n_i(t)}{t}, \qquad \nu_i = \limsup_{t\uparrow\infty} \frac{n_i(t)}{t},$$

for $i = 1, \dots, n$. These quantities are asymptotic bounds for the proportion of the time that the i-th system is used.

Assumption 3'. We assume that for some k with $1 \leq k \leq n$, we have $\|A_i\| < 1$ for $i \leq k$, $\|A_i\| \geq 1$ for $i > k$, and

$$\|A_1\|^{\mu_1} \cdots \|A_k\|^{\mu_k} \cdot \|A_{k+1}\|^{\nu_{k+1}} \cdots \|A_n\|^{\nu_n} < 1. \tag{5}$$

Theorem 1 can be reformulated for discrete-time systems in the following way:

Theorem 2. *Consider a switched linear discrete-time system of the form (4). Suppose that Assumptions 2' and 3' hold. Then the system is asymptotically stable.*

The proof is almost exactly the same as before.

If we consider the problem mentioned at the beginning of the section, and imagine a "solar panel" with many sections suffering unstable oscillations then the theorem will indicate the necessity of inserting a panel to extinguish the vibrations.

Giving that the solar panel is a periodical structure, such that the switching to its different components would be ruled by a travelling wave it follows that the switching signal $\sigma(t)$ would be given by a deterministic expression; hence, the engineer will have to choose the materials in the solar panel so that the assumptions of Theorem 2 hold and the travelling wave in it will extinguish.

A similar argument would allow one to deduce whether a wave would extinguish in a system governed by Schrödinger or Maxwell equations [17]

Remark 2. *It is straightforward for engineers to check whether Assumption 3' holds. For instance, using the Euclidean norm, one just calculates the norms*

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)}$$

as indicated earlier.

4. Hybrid Systems

When the system has both continuous and discrete subsystems we have a hybrid system.

A linear hybrid system can be described as follows. Starting at an integral time n in state $x(n)$, the system evolves as a continuous system governed by the equation

$$\dot{x}(t) = A_{\sigma_1(t)}x(t) \tag{6}$$

(where $\sigma_1 : [0, \infty) \to \{1, \cdots, n\}$ is a continuous-time switching signal) for one unit of time. At the end of that time unit, it reaches the state $x(n + 1-)$. Then it changes instantaneously according to

$$x(n + 1) = A_{\sigma_2(n)}x(n + 1-) \tag{7}$$

(where $\sigma_2 : \mathbb{N} \to \{1, \cdots, m\}$ is a discrete time switching signal).

These systems are more and more frequent in industry due to integration of continuous and discrete systems. The continuous system might have its origin in the flow or process of a factory, and the discrete one in the digital control of the diverse steps of the process. Hybrid systems give rise to the same problems formulated by Liberzon and Morse, that we have already mentioned formerly [13, 18, 19, 20]. We can deduce a theorem for these systems by combining theorems 1 and 2.

Theorem 3. *Consider a hybrid system given by (6) and (7). Suppose that the continuous subsystem (6) satisfies the conditions of Theorem 1 and the discrete subsystem (7) satisfies the conditions of Theorem 2. Then the hybrid system is asymptotically stable.*

Proof. The proof is straight-forward. It is enough to estimate the norms of the state $x(t)$ after a time t as before, and then to gather separately the terms corresponding to the continuous subsystem and to the discrete one. Then the estimates in the proofs of theorem 1 and 2 are respectively repeated for each group of terms. □

Remark 3. *If one of the subsystems has a bounded solution and the another one tends asymptotically to zero (because it satisfies its respective theorem) then the solution of hybrid system also tends asymptotically to zero. We will return to this remark later.*

5. Further Remarks

5.1. **Controllability.** The concept of controllability plays an important role in linear systems. A system is controllable if the state can be controlled by a switching signal [10, 21, 22]. The solution of this problem is critical for designers of switched linear systems, because it is usually essential that the system would always be under control.

The question: "*Does there exist a switching sequence by which the controllability is realised completely*" was first raised in [21]. A complete geometric characterisation for controllability of switched linear systems was established in [22], where sufficient and necessary conditions were established. In [23] a construction method for switching signal is provided. Later, it was tried to design switching signals in such a way that controllability was achieved with the number of switching as small as possible; Ji, Wang and Guo [24] established the relation between the number of switches and the dimension of the controllable space. These authors consider that although controllability conditions have been established, the behaviour of switching signal to get the controllability is not completely investigated. Within this frame are the theorems that we have shown is this paper; where information about switching signals that give controllability has been shown. Furthermore, complete controllability follows from the theorems, due to the fact that the initial input does not play any role in the proof of our theorems.

We have used an averaging idea in the formulation and proof of the theorems. This is similar to probabilistic analysis found in work that uses the concept of the average dwell-time [25], but in contrast to the average dwell-time approach we have worked with switched linear system that have unstable matrices. This is relevant information for engineers because they usually find systems of this type (switched linear systems ruled only by stable matrices are very limiting in practical problems).

5.2. **Feedback Stabilisation Problem.** If the switching signal of switched linear systems is not fixed, but depends on a parameter or can be designed by the engineers then the theorems proven in this paper allow one to design appropriate feedback control laws to make system stable. Let us show how to do that.

There exists a polynomial $p_i(t)$ whose degree is at most m, such that

$$\left\|e^{A_i t}\right\| \leq p_i(t) e^{\mu_i t}$$

where $\mu_i = \max\{\mathrm{Re}\lambda_i : \lambda_i \quad \text{eigenvalues} \quad \text{of} \quad A_i\}$. If we bound $|p_i(t)| \leq k_i$ in $[0, T_i]$ it follows that

$$\left\langle e^{A_i}\right\rangle \equiv \left[\prod_{i_j=1}^{n_i} \left\|e^{A_i t_{ij}}\right\|\right]^{\frac{1}{m_i}} \leq k_i e^{\mu_i \overline{t_i}}$$

where

$$\overline{t_i} = \frac{\sum_{i_j=1}^{m_i} t_{ij}}{m_i}$$

is the average time that system (1) stays in subsystem given by A_i.

Therefore

$$\prod_{i=1}^{n} \left\langle e^{A_i}\right\rangle \leq k e^{\sum_{i=1}^{n} \mu_i \overline{t_i}} \tag{8}$$

Thus, the time $\overline{t_i}$ can be deduced such that theorem (1) is satisfied and asymptotic stability is obtained. It is plain to see that $\sigma(t)$ will not be unique, because only the average time $\overline{t_i}$ is constrained, so engineers can choose any $\sigma(t)$ provided it is such that the average time $\overline{t_i}$ satisfies the assumptions of Theorem 1.

It does not matter whether the switched linear system has unstable matrices. The engineer must design the system in such a way it spends enough time (according to (8)) using stabilizing matrices, in order to control the unstable matrices.

For a discrete time system, theorem 2 shows that to get controllability one must use stabilizing matrices more than bad ones.

For a hybrid system we can control simultaneously the continuous and discrete subsystems according to what we have just said about these systems. Or we can control the continuous (discrete) subsystem if the discrete (continuous) has bounded solutions, due to Remark 3.

Therefore, in a continuous system, such as those engineers may find in a factory, they would be able to add a discrete system of the type described by Theorem 2 to get process controllability.

6. Conclusions

If a switched linear system has a switching signal such that a suitable weighted geometric average of stable subsystems dominates that of the unstable ones, then the solution of the system converges asymptotically to zero. The conditions are stated for continuous, discrete-time or hybrid systems, and allow engineers design a switching signal to get the controllability of the designed system.

The asymptotic average proportion of time in each subsystems determines the feedback needed enabling the engineer to control the system.

We would like to point out two facts, under the conditions of the theorems above:

i) The controllability of the system is obtained although it has unstable subsystems. That is important because systems with unstable subsystems are very frequent in engineering.

ii) If an engineer is looking for the controllability of a continuous system which is really hard to control then he can add a discrete-time system. In this new system, the controllability can be provided by the discrete-time subsystem. That is an advantage in current times, where digital systems overcome analogue ones, but where continuous systems are very widespread (think, for example, of an oil refinery).

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