CONSTRUCTING C^1 EXTENSIONS

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ABSTRACT

In [4], we proved that a function $f : X \to \mathbf{R}$ mapping a closed subset X of a C^k manifold M to $\mathbf{R}$ possesses a C^k extension to M if and only if the projection map $\pi : M \times \mathbf{R} \to M$ induces a bijection from the k–th order tangent star $\mathbf{Tan}^k(M \times \mathbf{R}, \mathrm{graph}(f))$ to $\mathbf{Tan}^k(M, X)$. Here it is shown that if $k = 1$ and the induced map is a bijection, then the extension can be explicitly constructed.

1. Introduction

In [4], we defined the k–th order *tangent star*, denoted $\mathbf{Tan}^k X$, of an arbitrary closed set X contained in a C^k manifold, M. We proved that a function $f : X \to \mathbf{R}$ mapping a closed subset X of a C^k manifold M to $\mathbf{R}$ possesses a C^k extension to M if and only if the projection map $\pi : M \times \mathbf{R} \to M$ induces a bijection from $\mathbf{Tan}^k(M \times \mathbf{R}, \mathrm{graph}(f))$ to $\mathbf{Tan}^k(M, X)$. Here it is shown that if $k = 1$ and the induced map is a bijection, then the extension can be explicitly constructed. In section 2, we recall the definitions introduced in [4], and provide some other prerequisites for the constructive proof, which is given in section 3.

2. Notation and definitions

First, we recall some of the definitions made in [4]: more details and some examples are provided in that paper.

Let $C^k(M)$ denote the Frechet algebra of all C^k real–valued functions on M, and let $C^k(M)^*$ denote its dual, which has a natural norm. To each closed $X \subset M$, we associate the ideal

$$I_k(X) = \{f \in C^k(M) : f|X = 0\}$$

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of functions that vanish on X, and we abbreviate

$$I_k(a) = I_k(\{a\}).$$

We define

$$\begin{aligned}
\mathbf{Tan}^k(M,a) &= C^k(M)^* \cap \left(I_k(a)^{k+1}\right)^{\perp},\\
\mathbf{Tan}^k(M,X,a) &= \mathbf{Tan}^k(M,a) \cap I_k(X)^{\perp},\\
\mathbf{Tan}^k(M) &= \bigcup_{a\in M} \mathbf{Tan}^k(M,a),\\
\mathbf{Tan}^k(M,X) &= \bigcup_{a\in M} \mathbf{Tan}^k(M,X,a).
\end{aligned}$$

In a natural way, we may identify $\mathbf{Tan}^k(M)$ with a subset of $\mathbf{Tan}^{k+1}(M)$, and get

$$\mathbf{Tan}^0(M) \subset \mathbf{Tan}^1(M) \subset \mathbf{Tan}^2(M) \cdots .$$

This allows us to define the *order* of an element ∂ of $\mathbf{Tan}^k(M)$ as the least i such that $\partial \in \mathbf{Tan}^i(M)$.

The module action of $C^k(M)$ on $C^k(M)^*$ restricts to a module action on each $\mathbf{Tan}^k(M,X,a)$, and, given local coordinates, determines a unique module action of a local algebra

$$\mathbf{R}[x]_{k,a} = \mathbf{R}[x_1,\ldots,x_d]/\langle (x-a)^i : |i| \le k+1\rangle.$$

Thus $\mathbf{Tan}^k(M,X,a)$ is a finite–dimensional real vector space and is a module over a finite–dimensional real algebra.

$\mathbf{Tan}^1(M,X,a)$ is the direct sum of $\mathbf{R}\delta_a$ (δ_a = evaluation at the point a) and the usual tangent space T_aM to M at a (thought of as a set of point derivations). If X is a submanifold near a, then $\mathbf{Tan}^1(M,X,a)$ is $\mathbf{R}\delta_a \oplus T_aX$, and $\mathbf{Tan}^k(M,X,a)$ is essentially the k–th order tangent space of Pohl [5].

If $f : (M,X,a) \to (N,Y,b)$ is a map of pointed pairs, then it induces a map $f_* : \mathbf{Tan}^k(M,X,a) \to \mathbf{Tan}^k(N,Y,b)$, and this association is functorial and the induced map does not increase order and is a module homomorphism.

The main theorem proved in [4] is as follows:

Theorem. *Let M be a C^k manifold, X be a closed subset of M, and $f : X \to \mathbf{R}$ be continuous. Let G denote the graph of f. Let $\pi : M \times \mathbf{R} \to M$ be the projection and denote the point $(a, f(a))$ by $\tilde{a}$. Then f has a C^k extension to M if and only if the map*

$$\pi_* : \mathbf{Tan}^k(M \times \mathbf{R}, G, \tilde{a}) \to \mathbf{Tan}^k(M,X,a)$$

is bijective for each $a \in X$.

In section 3 below, we show that if $k = 1$ and π_* is bijective, then the C^1 extension to M, whose existence is guaranteed by this theorem, may be explicitly constructed.

For $a \in \mathbf{R}^d$, we define the k–th order *Taylor map*

$$^{\to a} : \mathbf{Tan}^k \mathbf{R}^d \to \mathbf{Tan}^k(\mathbf{R}^d, a)$$

as the map, linear on rays, such that

$$(\partial^{\to a} - \partial)p(x) = 0$$

whenever $p(x) \in \mathbf{R}[x]$ has degree at most k. Putting it another way, if we define the map $^{b \leftarrow a}$ to be the map (a linear isomorphism) that makes the following diagram commute:

$$\begin{array}{ccc} & \mathbf{R}[x]_k & \\ \swarrow & & \searrow \\ \mathbf{R}[x]_{k,b} & \xleftarrow{\;b\leftarrow a\;} & \mathbf{R}[x]_{k,a} \end{array}$$

then the Taylor map from $\mathbf{Tan}^k(\mathbf{R}^d, b)$ to $\mathbf{Tan}^k(\mathbf{R}^d, a)$ is the adjoint of $^{b \leftarrow a}$. Explicitly,

$$\begin{aligned} \delta_b^{\to a} &= \delta_a + (b_1 - a_1)\frac{\partial}{\partial x_1}\bigg|_a + \cdots + (b_d - a_d)\frac{\partial}{\partial x_d}\bigg|_a \\ &\quad + \tfrac{1}{2}(b_1 - a_1)^2 \frac{\partial^2}{\partial x_1^2}\bigg|_a + \cdots \\ &\quad \cdots + (b_d - a_d)^k \frac{\partial^k}{\partial x_d^k}\bigg|_a , \\ \frac{\partial}{\partial x_1}\bigg|_b^{\to a} &= \frac{\partial}{\partial x_1}\bigg|_a + (b_1 - a_1)\frac{\partial^2}{\partial x_1^2}\bigg|_a + \cdots \\ &\quad \cdots + (b_d - a_d)\frac{\partial^k}{\partial x_1 \partial x_d^{k-1}}\bigg|_a \\ &\quad \cdots \\ \frac{\partial^k}{\partial x_1^k}\bigg|_b^{\to a} &= \frac{\partial^k}{\partial x_1^k}\bigg|_a , \\ &\quad \cdots . \end{aligned}$$

Note that this map depends on k.

One should think of $\partial^{\to a}$ as the *nearest* tangent at a to the tangent ∂, in a certain sense (— but not in the sense of the metric of $\mathcal{C}^{k*}$). Observe that for $\partial \in \mathbf{Tan}^k(\mathbf{R}^d, b)$, the weak–star limit as $a \to b$ of the Taylor–mapped tangents $\partial^{\to a}$ is ∂, and if ∂ has order $j < k$, then for each function $f \in \mathcal{C}^k$, the error will be $\mathrm{o}(|a-b|^{k-j})$.

It is obvious that in general

$$\partial^{\to b \to a} = \partial^{\to a}.$$

3. Constructing the extension

Let M be a C^k manifold, X be a closed subset of M, and $f : X \to \mathbf{R}$ be continuous. Let G denote the graph of f. Let $\pi : M \times \mathbf{R} \to M$ be the projection and for $x \in X$ denote the point $(x, f(x))$ by $\tilde{x}$. In this section we show that if the map

$$\pi_* : \mathbf{Tan}^k(M \times \mathbf{R}, G, \tilde{a}) \to \mathbf{Tan}^k(M, X, a)$$

is bijective for each $a \in X$, then we can construct a C^k extension of f to M.

It is shown in Section 4 of [4] that we need only consider the case $M = \mathbf{R}^d$. We abbreviate $\mathbf{Tan}^k(\mathbf{R}^d, X, a)$ to $\mathbf{Tan}^k(X, a)$, and $\mathbf{Tan}^k(\mathbf{R}^{d+1}, G, \widetilde{a})$ to $\mathbf{Tan}^k(G, \widetilde{a})$.

By hypothesis, each $\partial \in \mathbf{Tan}^1(X)$ has a unique $\widetilde{\partial} \in \mathbf{Tan}^1(G)$ such that $\pi_*\widetilde{\partial} = \partial$. For instance,

$$\widetilde{\delta}_a = \delta_{\widetilde{a}}, \quad \forall a \in X.$$

Define the 1-jet $\widetilde{f}\colon \mathbf{Tan}^1(X) \to \mathbf{R}$ by

$$\langle \partial, \widetilde{f} \rangle = \widetilde{\partial} y.$$

(This definition is motivated by the fact that if f were a C^1-function, then

$$\widetilde{\partial} h(x, y) = \partial h(x, f(x)),$$

so

$$\widetilde{\partial} y = \partial f.)$$

What we must do to prove the result is to extend $\widetilde{f}$ to a suitable 1-jet on $\mathbf{Tan}^1(\mathbf{R}^d)$.

Let $\partial(a, u)$ denote the tangent $g \mapsto \langle u, \nabla g(a) \rangle$, where $a \in \mathbf{R}^d$ and $u \in \mathbf{R}^d$.

Let

$$X_j = \{a \in \mathbf{R}^d \colon \dim \mathbf{Tan}^1(X, a) \ge j\}.$$

Then each X_j is closed, $X_0 = \mathbf{R}^d$, $X_1 = X$, X_2 is the set of accumulation points of X, and $X_{d+1} \subseteq X_d$.

We may assume that $X_{d+1} \neq \emptyset$, for otherwise we adjoin a remote closed ball B to X and define $f \equiv 0$ on B.

Let $D_{d+1} = \mathbf{Tan}^1(X)$.

We shall construct the extension of $\widetilde{f}$ by first extending it to the closed star

$$\begin{aligned} D_d &= \mathbf{Tan}^1(X) \cup \{\partial(a,u) \colon a \in X_d, u \in \mathbf{R}^d\} \\ &= \mathbf{Tan}^1(X) \cup \mathrm{pt}^{-1}(X_d), \end{aligned}$$

then extending it to

$$D_{d-1} = \mathbf{Tan}^1(X) \cup \mathrm{pt}^{-1}(X_{d-1}),$$

and so on.

Each step in the construction is like the proof of Whitney's extension theorem, with an additional complication.

We say that a closed star $D \subset \mathbf{Tan}^1(\mathbf{R}^d)$ is *full on* Y if

$$Y = \{a \in \mathbf{R}^d \colon \mathrm{dim} D(a) = d+1\}.$$

Thus the star D_j is full on X_j, for $j = d+1, \ldots, 0$.

To extend $\widetilde{f}$ from D_{j+1} to D_j, we have to define $\widetilde{f}\partial$ for $\partial \in \mathbf{Tan}^1(\mathbf{R}^1, a) \sim \mathbf{Tan}^1(X, a)$ with $a \in X_j \sim X_{j+1}$. We want to do this in such a way that the following properties hold:

(P1) $\widetilde{f}$ is a 1-jet on D_j, that is, $\widetilde{f}$ is linear on rays;
(P2) $\widetilde{f}\partial(a_n, u_n) \to \widetilde{f}\partial(a,u)$ whenever $\partial(a_n,u_n) \in D_j$ and $\partial(a_n,u_n) \to \partial(a,u)$ weak-star;
(P3) for each compact $K \subset \mathrm{pt}(D_j)$, $\forall \epsilon > 0$ $\exists \delta > 0$ such that $\forall a, x, y \in K$ and $\forall u \in \mathbf{R}^d$ such that $x \neq y$, $|x-a| < \delta$, $|y-a| < \delta$, $\left|\frac{x-y}{|x-y|} - u\right| < \delta$, and $\partial(a,u) \in D_j$ we have

$$\left| \frac{\widetilde{f}(\delta_x) - \widetilde{f}(\delta_y)}{|x-y|} - \widetilde{f}\partial(a,u) \right| < \epsilon.$$

Note that for $j > 0$ we may write $f(x)$ instead of $\langle \delta_x, \widetilde{f} \rangle$. Plainly, once we have the above properties on $D_0 = \mathbf{Tan}^1(\mathbf{R}^d)$, we are done, and $x \mapsto \langle \delta_x, \widetilde{f} \rangle$ is the desired extension.

The standard Whitney construction achieves the last step, from D_1 to D_0.

To begin, we must demonstrate that properties (P1), (P2) and (P3) hold on D_{d+1}. This we do in a series of steps.

Claim 1. *Let $\mu_n \in span\mathbf{Tan}^1(G)$, $\|\mu_n\|_{\mathcal{C}^{1*}} \le M$, and*

$$\pi_\sharp \mu_n \to \partial \in \mathbf{Tan}^1(X),$$

weak-star in $\mathcal{C}^1(\mathbf{R}^d)^*$. Then $\mu_n \to \widetilde{\partial}$, weak-star in $\mathcal{C}^1(\mathbf{R}^{d+1})^*$.

Proof. Let μ be any weak-star accumulation point of $\{\mu_n\}$. Choose a net $\{\mu_{n_\alpha}\}$ such that $\mu_{n_\alpha} \to \mu$. Then $\pi_\sharp \mu_{n_\alpha} \to \pi_\sharp \mu$, since $\pi_\sharp$ is weak-star to weak-star continuous. Thus $\pi_\sharp \mu = \partial$, so $\mu = \widetilde{\partial}$. Consequently, the intersection of the weak-star compact convex sets

$$\text{weak-star clos}\Big(\text{convex hull}\Big(\{\mu_n)\}_{n \ge N}\Big)\Big)$$

$(N = 1, 2, 3, \dots)$ is $\Big\{\widetilde{\partial}\Big\}$, so $\mu_n \to \widetilde{\partial}$, weak-star. QED

Claim 2. *$f \in \mathrm{Lip}(1, X)_{loc}$, that is, $f \in \mathrm{Lip}(1, K)$ for each compact $K \subset X$.*

Proof. Otherwise, there exist x_n, $y_n \in K$ such that $x_n \ne y_n$ and

$$\frac{f(x_n) - f(y_n)}{|x_n - y_n|} \uparrow +\infty.$$

We may assume that $x_n \to a$ and $y_n \to a$ for some $a \in K$.

Consider

$$\mu_n = \frac{\delta_{\widetilde{x_n}} - \delta_{\widetilde{y_n}}}{f(x_n) - f(y_n)}.$$

Clearly, $\mu_n \to \frac{\partial}{\partial y}|_a$ weak-star. Since $\mu_n \in I_1(G)^\perp$, this gives $\frac{\partial}{\partial y}|_a \in \mathbf{Tan}^1(G)$, which contradicts the injectivity of π_*. QED

Claim 3. *If $\partial \in \mathbf{Tan}^1(X, a)$, and $g \in \mathcal{C}^1(\mathbf{R}^{d+1})$, then*

$$\widetilde{\partial} g(x, y) = (\partial 1) g(\widetilde{a}) + \{\partial(x_1 - a_1)\}\frac{\partial g}{\partial x_1}(\widetilde{a}) + \cdots + \{\partial(x_d - a_d)\}\frac{\partial g}{\partial x_d}(\widetilde{a})$$
$$+ \widetilde{f}(\partial)\frac{\partial g}{\partial y}(\widetilde{a}) - (\partial 1) f(a)\frac{\partial g}{\partial y}(\widetilde{a}).$$

In particular, we have

$$\widetilde{\partial(a, u)} = u_1 \frac{\partial g}{\partial x_1}(\widetilde{a}) + \cdots + u_d \frac{\partial g}{\partial x_d}(\widetilde{a}) + \widetilde{f}\partial(a, u)\frac{\partial g}{\partial y}(\widetilde{a}),$$

whenever $\partial(a, u) \in \mathbf{Tan}^1(X)$.

Proof.

$$g(x, y) = g(\widetilde{a}) + (x_1 - a_1)\frac{\partial g}{\partial x_1}(\widetilde{a}) + \cdots + (x_d - a_d)\frac{\partial g}{\partial x_d}(\widetilde{a})$$
$$+ (y - f(a))\frac{\partial g}{\partial y}(\widetilde{a}) + h(x, y)$$

where $h \in \mathcal{C}^1(R^{d+1})$ and $\nabla h(\widetilde{a}) = 0$. Thus, since $\widetilde{\partial} h = 0$, the claim follows. QED

Claim 4. *For each compact $K \subset X$ there exists $M > 0$ such that*

$$|\widetilde{f}\partial(a,u)| \leq M$$

whenever $a \in K$, $|u| \leq 1$ and $\partial(a,u) \in \mathbf{Tan}^1(X)$.

Proof. Otherwise there exist $a_n \in K$ and $u_n \in \mathbf{R}^d$ such that $\partial(a_n, u_n) \in \mathbf{Tan}^1(X)$ and

$$\widetilde{f}\partial(a_n, u_n) \uparrow +\infty.$$

We may assume that $a_n \to a$ and $u_n \to u$, whence $\partial(a,u) \in \mathbf{Tan}^1(X)$. Consider

$$\mu_n = \frac{\partial\widetilde{(a_n, u_n)}}{\widetilde{f}\partial(a_n, u_n)}.$$

By Claim 3, for $g \in \mathcal{C}^1(\mathbf{R}^{d+1})$,

$$\mu_n g \to \frac{\partial g}{\partial y}(\widetilde{a}),$$

hence $\frac{\partial}{\partial y}|_{\widetilde{a}} \in \mathbf{Tan}^1(G)$, which is impossible. QED

Claim 5. *For each compact $K \subset X$ there exists $M > 0$ such that*

$$\|\partial\widetilde{(a,u)}\|_{\mathcal{C}^{k*}} \leq M$$

whenever $a \in K$, $|u| \leq 1$ and $\partial(a,u) \in \mathbf{Tan}^1(X)$.

Proof. We have, by Claim 3,

$$\|\partial\widetilde{(a,u)}\|_{\mathcal{C}^{k*}} \leq |u_1| + \cdots + |u_j| + |\widetilde{f}\partial(a,u)|,$$

so the result follows from Claim 4. QED

Claim 6. *Let $\partial(a,u) \in \mathbf{Tan}^1(X)$ be nonzero. Then*

$$\lim_{\substack{(b,v)\to(a,u)\\ \partial(b,v)\in\mathbf{Tan}^1(X)}} \widetilde{f}\partial(b,v) = \widetilde{f}\partial(a,u).$$

Proof. Note that away from 0, $\mathbf{Tan}^1(X)$ is locally metrisable.

Let $\partial(b_n, v_n) \in \mathbf{Tan}^1(X)$ and $(b_n, v_n) \to (a,u)$. Then by Claim 5 and Claim 1 we get

$$\partial\widetilde{(b_n, v_n)} \to \partial\widetilde{(a,u)},$$

weak-star, hence

$$\widetilde{f}\partial(b_n, v_n) = \partial\widetilde{(b_n, v_n)}y \to \partial\widetilde{(a, u)}y = \widetilde{f}\partial(a, u).$$

QED

Claim 7. *Let $K \subset X$ be compact and $\epsilon > 0$. Then there exists $\delta > 0$ such that for each $a, x, y \in K$ and each $u \in \mathbf{R}^d$ such that $x \neq y$, $|x - a| < \delta$, $|y - a| < \delta$, $\left|\frac{x-y}{|x-y|} - u\right| < \delta$, and $\partial(a, u) \in \mathbf{Tan}^1(X)$, we have that*

$$\left|\frac{f(x) - f(y)}{|x - y|} - \widetilde{f}\partial(a, u)\right| < \epsilon.$$

Proof. Otherwise there exist a_n, x_n, $y_n \in K$ and $u_n \in \mathbf{R}^d$ such that $x_n \neq y_n$, $x_n - a_n \to 0$, $y_n - a_n \to 0$, $\frac{x_n-y_n}{|x_n-y_n|} - u_n \to 0$, $\partial(a_n, u_n) \in \mathbf{Tan}^1(X)$ and

$$\left|\frac{f(x_n) - f(y_n)}{|x_n - y_n|} - \widetilde{f}\partial(a_n, u_n)\right| \geq \epsilon.$$

We may assume that $a_n \to a \in K$ and that $u_n \to u \in \mathbf{S}^1$. Then $x_n \to a$, $y_n \to a$ and

$$\frac{x_n - y_n}{|x_n - y_n|} \to u.$$

It follows that

$$\frac{\delta_{x_n} - \delta_{y_n}}{|x_n - y_n|} \to \partial(a, u)$$

weak-star in $\mathcal{C}^1(\mathbf{R}^d)^*$. By Claim 2, the functionals

$$\mu_n = \frac{\delta_{\widetilde{x_n}} - \delta_{\widetilde{y_n}}}{|x_n - y_n|} \in \mathrm{span}\mathbf{Tan}^1(G)$$

are norm-bounded in $\mathcal{C}^1(\mathbf{R}^{d+1})^*$, hence, by Claim 1,

$$\mu_n \to \partial\widetilde{(a, u)}$$

weak-star in $\mathcal{C}^1(\mathbf{R}^{d+1})^*$. Thus

$$\frac{f(x_n) - f(y_n)}{|x_n - y_n|} = \mu_n y \to \partial\widetilde{(a, u)}y = \widetilde{f}\partial(a, u).$$

Also, by Claim 6,

$$\widetilde{f}\partial(a_n, U_n) \to \widetilde{f}\partial(a, u),$$

and we get a contradiction. QED

At this stage, we are ready to commence the induction, since we have now shown that D_{d+1} has properties (P1), (P2) and (P3).

Suppose now that $\widetilde{f}$ is defined on D_{j+1} and has properties (P1), (P2) and (P3) there. We will show how to extend $\widetilde{f}$ to D_j. First we consider $j > 0$.

If $D_j \sim D_{j+1} = \emptyset$ (that is, $X_j \sim X_{j+1} = \emptyset$), then there is nothing to do, so suppose that $X_j \sim X_{j+1} \neq \emptyset$.

We put an inner product on each ray $\mathbf{Tan}^1(\mathbf{R}^d, a)$ by defining

$$\left\langle \alpha_0 + \alpha_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_d \frac{\partial}{\partial x_d}, \beta_0 + \beta_1 \frac{\partial}{\partial x_1} + \cdots + \beta_d \frac{\partial}{\partial x_d} \right\rangle = \alpha_0 \beta_0 + \cdots + \alpha_d \beta_d.$$

Let P_a denote the orthogonal projection of $\mathbf{Tan}^1(\mathbf{R}^d, a)$ on $D_{j+1}(a)$ with respect to this inner product, and let N_a denote $1 - P_a$, the projection on the orthogonal complement of $D_{j+1}(a)$ in $\mathbf{Tan}^1(\mathbf{R}^d, a)$.

We take a Whitney system for $X_j \sim X_{j+1}$, that is, a family $\{Q_n\}$ of cubes and a corresponding family $\{\phi_n\}$ of functions such that

(a) $\kappa_1 \cdot \operatorname{dist}(Q_n, X_{j+1}) < \operatorname{side} Q_n < (d+1) \cdot \operatorname{dist}(Q_n, X_{j+1})$,
(b) no point belongs to more than κ_2 of the Q_n,
(c) $X_j \sim X_{j+1} \subset \bigcup_{n=1}^{\infty} Q_n$,

and such that $\phi_n \in \mathcal{C}^\infty_{\mathrm{cs}}$ with $\operatorname{spt}\phi_n \subset Q_n$, $\sum \phi_n = 1$ on $X_j \sim X_{j+1}$, $0 \leq \phi_n \leq 1$, and $(\operatorname{side} Q_n)|\nabla \phi_n| \leq \kappa_3$.

Here $\kappa_1, \kappa_2, \kappa_3$ are constants that depend only on d, and by the distance between two sets we mean the infimum of the distances of pairs of points, one from each set.

For each n, let c_n be a closest point of X_{j+1} to Q_n.

Now, for $a \in X_j \sim X_{j+1}$ and $\partial \in \mathbf{Tan}^1(\mathbf{R}^d, a)$, we define

$$\widetilde{f}\partial = \widetilde{f}(P_a \partial) + \sum_{n=1}^{+\infty} \phi_n(a) \widetilde{f}\left((N_a \partial)^{\to c_n}\right).$$

This extends the previous $\widetilde{f}$, since $P_a \partial = \partial$ for $\partial \in D_{j+1}$. Here, the Taylor maps are to be understood as C^1 Taylor maps.

Plainly, $\widetilde{f}$ is linear on rays, since the projections P_a,N_a, the Taylor maps ${}^{\to c_n}$, and the previous $\widetilde{f}$ are all linear.

To check the continuity of $\widetilde{f}\partial(a,u)$ on D_j we must consider $\partial(a_n,u_n) \to \partial(a,u)$ weak-star, and there are three cases:

1^0. $\partial(a_n,u_n) \in D_{j+1}$, $\partial(a,u) \in D_{j+1}$;
2^0. $\partial(a_n,u_n) \in D_j \sim D_{j+1}, a \in X_{j+1}$;
3^0. $\partial(a_n,u_n) \in D_j \sim D_{j+1}, a \in X_j$.

Case 1^0: We have $\widetilde{f}\partial(a_n,u_n) \to \widetilde{f}\partial(a,u)$ by the induction hypothesis.

Case 2^0: We have

$$\widetilde{f}\partial(a_n,u_n) = \widetilde{f}P_{a_n}\partial(a_n,u_n) + \sum_{m=1}^{+\infty} \phi_m(a_n)\widetilde{f}\left((N_{a_n}\partial(a_n,u_n))^{\to c_m}\right).$$

Let P'_a denote the projection on $\mathbf{R}^d$ corresponding to P_a, that is,

$$P_a\partial(a,u) = \partial(a, P'_a u), \quad \forall u \in \mathbf{R}^d.$$

Let $v_n = P'_{a_n}u_n$. Then

$$\begin{aligned} P_{a_n}\partial(a_n,u_n) &= \partial(a_n,v_n) \\ N_{a_n}\partial(a_n,u_n) &= \partial(a_n,u_n-v_n) \\ \text{and} \qquad (N_{a_n}\partial(a_n,u_n))^{\to c_m} &= \partial(c_m,u_n-v_n). \end{aligned}$$

Then $\operatorname{dist}(a_n,X_{j+1}) \to 0$ as $n \uparrow \infty$, so

$$\sup\{|c_m-a| \colon \phi_m(a_n) \neq 0\} \to 0,$$

and hence

$$\begin{aligned} \widetilde{f}\partial(a_n,u_n) &= \widetilde{f}\partial(a_n,v_n) + \sum_{m=1}^{+\infty} \phi_m(a_n)\widetilde{f}\partial(c_m,u_n-v_n) \\ &= \widetilde{f}\partial(a,v_n) + \sum_{m=1}^{+\infty} \phi_m(a_n)\widetilde{f}\partial(a,u_n-v_n) + \mathrm{o}(1) \\ &= \widetilde{f}\partial(a,u_n) + \mathrm{o}(1), \quad \text{since} \sum_{m=1}^{+\infty} \phi_m(a_n) = 1 \\ &= \widetilde{f}\partial(a,u) + \mathrm{o}(1). \end{aligned}$$

Case 3^0: Consider the sequence of orthogonal projections P'_{a_n} on $\mathbf{R}^d$. Each limit point Q of $\{P'_{a_n}\}$ is a rank $j-1$ orthogonal projection, and $\partial(a,u) \in \mathbf{Tan}^1(X,a)$ $\forall u \in \operatorname{im}Q$. Since

$$\dim\{u : \partial(a,u) \in \mathbf{Tan}^1(X,a)\} = j-1,$$

Q is unique, so $\{P'_{a_n}\}$ converges, and indeed $P'_{a_n} \to P'_a$. Thus, bearing in mind that all but a finite number of ϕ_m are zero on all a_n's,

$$
\begin{aligned}
P_{a_n}\partial(a_n, u_n) &= \partial(a_n, P'_{a_n}u) \\
&= \partial(a, P'_a u) + \mathrm{o}(1), \\
N_{a_n}\partial(a_n, u_n) &= \partial(a, u - P'_a u) + \mathrm{o}(1), \\
\phi_m(a_n) &= \phi_m(a) + \mathrm{o}(1), \\
\widetilde{f}\partial(a_n, u_n) &= \widetilde{f}\partial(a, P'_a u) + \sum_{m=1}^{+\infty} \phi_m(a)\widetilde{f}\partial(c_m, u - P'_a u) + \mathrm{o}(1) \\
&= \widetilde{f}\partial(a, u) + \mathrm{o}(1).
\end{aligned}
$$

If the property (P3) fails, then there exists compact $K \subset \mathrm{pt}D_j$, $\epsilon > 0$, $a_n, x_n, y_n \in K$ and $u_n \in \mathbf{R}^d$ such that $x_n \neq y_n$, $x_n - a_n \to 0$, $y_n - a_n \to 0$,

$$
\frac{x_n - y_n}{|x_n - y_n|} - u_n \to 0, \qquad \partial(a_n, u_n) \in D_j
$$

and

$$
\left| \frac{\widetilde{f}(\delta_{x_n}) - \widetilde{f}(\delta_{y_n})}{|x_n - y_n|} - \widetilde{f}\partial(a_n, u_n) \right| \geq \epsilon.
$$

We may assume that $a_n \to a$, $u_n \to u$, and hence that $x_n \to a$, $y_n \to a$, $\frac{x_n - y_n}{|x_n - y_n|} \to u$, $\partial(a, u) \in D_j$ and (using property (P2))

$$
\left| \frac{\widetilde{f}(\delta_{x_n}) - \widetilde{f}(\delta_{y_n})}{|x_n - y_n|} - \widetilde{f}\partial(a, u) \right| \geq \frac{\epsilon}{2}.
$$

Since $j > 0$, then $\mathrm{pt}D_j = X$, so $\partial(a, u) \in \mathbf{Tan}^1(X)$ and Claim 7 gives a contradiction. Thus (P3) holds.

It remains to consider the last case, $j = 0$. The extension formula is then the classical Whitney formula

$$
\begin{aligned}
\widetilde{f}\delta_a &= \sum_{n=1}^{+\infty} \phi_n(a)\widetilde{f}(\overrightarrow{\delta_a}^{c_n}), \\
\widetilde{f}\delta(a, u) &= \partial(a, u)(x \mapsto \widetilde{f}\delta_x).
\end{aligned}
$$

The $\widetilde{f}$ that we begin with is a full set of Whitney data on D_1, and the proof in this case is simpler and well-known. (It is at this stage that the condition $\mathrm{diam}Q_n \cdot |\nabla\phi_n| \leq \kappa$ becomes important.) We omit the details. QED

We note a corollary of the proof.

Corollary. *If* $\partial \in \mathbf{Tan}^1 X$ *then there exist* $\mu_{nm} \in span\{\delta_a : a \in X\}$ *such that*

$$\partial = \textit{weak–star} \lim_{n\uparrow\infty} \textit{weak–star} \lim_{m\uparrow\infty} \mu_{nm}.$$

More specifically, if $\partial(a,u) \in \mathbf{Tan}^1(X)$*, then there exist* $x_{nmi}, y_{nmi} \in X$ *with* $x_{nmi} \neq y_{nmi}$*,* $u_{n1}, \ldots, u_{nd} \in \mathbf{R}^d$*, and* $\lambda_{ni} \in R$ *such that*

$$\frac{x_{nmi} - y_{nmi}}{|x_{nmi} - y_{nmi}|} \to u_{ni} \quad \textit{as} \quad m \uparrow +\infty,$$

$$\lambda_{n1}u_{n1} + \cdots + \lambda_{nd}u_{nd} \to u \quad \textit{as} \quad n \uparrow +\infty.$$

Proof. Let $T(X)$ be the weak-star closure in $\mathbf{Tan}^1(\mathbf{R}^d)$ of the star

$$\{\alpha\delta_a + \partial(a,u) : \alpha \in \mathbf{R}, u \in D(X,a)\}.$$

We may carry out the whole proof with $T(X)$ in place of $\mathbf{Tan}^1(X)$.

We claim that $T(X) = \mathbf{Tan}^1(X)$. If not, choose $(a,u) \in \mathbf{Tan}^1(X)$ with $\partial(a,u)$ orthogonal to $T(X)(a)$ with respect to the inner product on $\mathbf{Tan}^1(\mathbf{R}^d, a)$.

With $X_j = \{a \in X : \dim T(X)(a) \geq j\}$, choose j such that $a \in X_j \sim X_{j+1}$.

Take $f \in \mathcal{C}^1$ with $f(x) = \langle u, x \rangle$ on X_{j+1} and $f = 0$ near a. The extension formula, applied to the restriction $f|X$, gives

$$\widetilde{f}\partial(a,u) = 0 + \sum_n \phi_n(a).1 = 1.$$

Denote the extension by f^*. Then $f - f^* \in \mathcal{C}^1$ and vanishes on X and has

$$\partial(a,u)(f - f^*) = -1 \neq 0.$$

Thus $\partial(a,u) \notin \mathbf{Tan}^1(X)$, a contradiction. QED

Remark. A version of this corollary was given in an earlier paper by the first author ([2], p.320), but the proof provided there was rather terse. That paper and [3] provides a couple of explicit C^1 extension theorems, but the methods used there have no hope of dealing with C^2 extensions. We have used the methods based on $\mathbf{Tan}^k$ to work out explicit constructions for the case $k = 2$, $d = 1$, that is, C^2 extensions in 1 dimension. These will appear elsewhere.

Merrien [1] gave a constructive condition for the existence of a C^k extension in the one–dimensional case. His condition involved the uniform continuity of a constructively–defined divided difference $f[x_0, \ldots, x_d]$ on the $(k+1)$–st symmetric product

$X \times \cdots \times X$. This is less straightforward to verify in examples than the condition based on $\mathbf{Tan}^k$, since the latter condition involves only the examination of a finite–dimensional vector space at each point.

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