SOBOLEV SPACE ANALYTIC CAPACITIES

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Abstract

Analytic capacities are set functions defined on the family of compact subsets of the plane which may be used in the study of removable singularities, boundary smoothness, and approximation of analytic functions. The purpose of the present paper is to investigate the analytic capacities associated to Sobolev spaces. For technical reasons, we do not work with the usual Sobolev spaces $W^{k,p}$. Instead, we work with locally–equivalent spaces, denoted $W^{k,p}_\infty$.

We denote the $\bar\partial$–$W^{k,p}_\infty$–cap by $\gamma_{k,p}$, and we abbreviate $\gamma_{1,p}$ to γ_p. We denote the area of the set E by $|E|$.

For each $p \in [1,\infty]$, the null-sets of γ_p are precisely the sets of zero area. For $1 < p \leq +\infty$, there exists $\kappa(p) > 1$ such that

$$\kappa^{-1} \cdot |E|^{1/q} \leq \gamma(E) \leq \kappa \cdot |E|^{1/q},$$

for each compact $E \subset \mathbf{C}$ of diameter at most 1. Here q is the conjugate index: $q = p/(p-1)$, and $|E|$ denotes the area of E. The particular case $p = +\infty$ was already known, and is due to Nguyen.

The capacity γ_1 remains obscure. We investigate the relationship between γ_1 and the function $\ell(E) = 1/\log(1/|E|)$. We show that $\ell(E) = \mathrm{O}\left(\gamma_1(E)\right)$ whenever the compact set $E \subset \mathbf{C}$ has diameter at most 1. But we also show that $\gamma_1(E) \neq \mathrm{O}\left(\ell(E)\right)$.

For $k \geq 3$, or $k = 2$ and $p > 2$, we have $\gamma_{k,p}(E) = 0$ if and only if the interior of E is empty. In this case, we present some estimates for $\gamma_{k,p}$ in terms of the function $\mathrm{dist}(z, \mathbf{C} \sim E)$.

For $k = 2$ and $1 < p \leq 2$, $\gamma_{k,p}(E) = 0$ if and only if E has empty interior with respect to an appropriate fine topology, which we call the p–topology. The problem of giving a real–variable description of the null–sets of $\gamma_{2,1}$ is open.

1. Notation and Preliminaries.

(1.1) Throughout the paper, A stands for a positive absolute constant, which may be different at each occurrence.

By L^p we mean the usual space $L^p(\mathbf{C}, dxdy)$, of complex-valued measurable p–th power integrable functions on the plane (or, when $p = \infty$, essentially–bounded functions). The Sobolev space $W^{k,p}$ consists of those distributions f such that f and all its distributional partial derivatives up to and including order k are (representable by integration against) L^p functions. These are all Banach spaces, with standard norms [1, 5].

We denote the area of a set $E \subset \mathbf{C}$ by $|E|$.

We are interested in local questions about functions belonging to these Sobolev spaces, and in particular functions that are analytic on some open set $U \subset \mathbf{C}$. The kind of questions of interest relate to boundary smoothness properties, removable singularities, and approximation. For such questions, analytic capacities (defined below) are useful. These capacities are non–negative set functions, associated to the function spaces. However, it happens that some Sobolev spaces have identically–zero analytic capacity. This comes about for non–local reasons, having to do with the behaviour of the spaces near infinity. For this reason, we replace the Sobolev spaces by spaces $W^{k,p}_\infty$ that are, in a precise sense, locally–equivalent to the originals. These spaces are constructed by an application of a standard method called the F_∞–construction [9, p. 193, or 8, p.98]. Adapted to the present case, it goes as follows:

(1.2) We use C^∞ to denote the space of infinitely–differentiable complex–valued functions on the plane, $C^\infty_{\rm cs}$ for the space of test functions, $C^{\infty\prime}_{\rm cs}$ for the space of distributions, and $C^{\infty\prime}$ for the space of distributions with compact support. We use $\langle\phi, f\rangle$ to denote the action of the distribution f on the smooth function ϕ. In the case where f is (representable by) an integrable function, this means that

$$\langle\phi, f\rangle = \int \phi f \, dxdy.$$

We define

$$L^p_{\rm loc} = C^\infty \cdot L^p = \{f \in C^{\infty\prime}_{\rm cs} : \phi f \in L^p, \ \forall \phi \in C^\infty_{\rm cs}\}.$$

$L^p_{\rm loc}$ is a Frechet space, with topology defined by the seminorms

$$f \mapsto \|f\|_{L^p(X)}, \quad X \text{ compact}.$$

L^p_∞ is the space of those $f \in L^p_{\rm loc}$ such that

$$\|f\|_{L^p(\mathbf{B}(a,1))} \to 0 \text{ as } a \to \infty,$$

and it is normed by

$$\|f\|_{L^p_\infty} = \sup_{a \in \mathbf{C}} \|f\|_{L^p(\mathbf{B}(a,1))}.$$

Next, we define $W^{k,p}_{\text{loc}}$ to be the set of all f in L^p_{loc} with the property that each distributional derivative, $\partial^\alpha f$, of f is also in L^p_{loc} for every multi-index α such that $|\alpha| \le k$.

We define

$$\|f\|_{W^{k,p}_\infty} = \sup_{a\in\mathbf{C}} \left(\sum_{1\le|i|\le k} \|\partial^i f\|_{L^p(\mathbf{B}(a,1))} \right),$$

and we set

$$W^{k,p}_\infty = \left\{ f \in W^{k,p}_{\text{loc}} : \|f\|_{W^{k,p}_\infty} < \infty \right\}.$$

(1.3) It is worth noting that $\|f\|_{W^{1,p}_\infty}$ is equivalent (up to universally–constant multiplicative bounds) to

$$\|\partial f\|_{L^p_\infty} + \|\overline{\partial} f\|_{L^p_\infty},$$

where ∂ and $\bar\partial$ denote the differential operators

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$
$$\bar\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We shall normally work with this form of the norm. We recall Weyl's Lemma, which states that a distribution f is representable by an analytic function when restricted to some given open set U if and only if $\bar\partial f = 0$ (in the sense of distributions) on U.

(1.4) When a function f is analytic near infinity and zero at ∞, with Laurent expansion

$$f(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots,$$

then we denote a_1 by $f'(\infty)$. If $f \in W^{k,p}_\infty$ and $\overline{\partial} f = 0$ off a compact set E, then one finds that

$$f'(\infty) = \frac{1}{\pi}\int_E \overline{\partial} f \, dxdy.$$

In fact, if $\phi \in C^\infty_{\text{cs}}$ has $\phi = 1$ near E, then an application of Pompeiu's formula shows that

$$f'(\infty) = \frac{-1}{\pi}\langle \bar\partial\phi, f\rangle = \frac{1}{\pi}\langle \phi, \bar\partial f\rangle,$$

which immediately yields the stated result.

(1.5) We say that a distribution f is $\overline{\partial} - W^{k,p}_\infty$*−admissible* for the compact set $E \in \mathbf{C}$ if $f \in W^{k,p}_\infty$, f is analytic on $\mathbf{C} \sim E$, $f(\infty) = 0$ and $\|f\|_{W^{k,p}_\infty} \le 1$.

We then define the $\bar\partial$–$W^{k,p}_\infty$ *capacity* (or *analytic* $W^{k,p}_\infty$ *capacity*) of a compact set E to be

$$\overline{\partial} - W^{k,p}_\infty - \text{cap}(E) = \sup\left\{ |f'(\infty)| : f \text{ is } \overline{\partial} - W^{k,p}_\infty - \text{admissible for } E \right\}.$$

For the purposes of this paper, we denote the $\bar\partial$–$W^{k,p}_\infty$–cap by $\gamma_{k,p}$, and we abbreviate $\gamma_{1,p}$ to γ_p.

The $\gamma_{0,p}$ capacities (associated to L^p) have been extensively studied, and we have nothing to say about them in this paper, nor shall we use any results about them. We study only the cases $k \geq 1$.

By the way, the reason we use $W^{k,p}_\infty$ rather than $W^{k,p}$ is that if $W^{k,p}$ analytic capacities are defined in the same way as above, then they are sometimes identically zero. The $W^{k,p}_\infty$ analytic capacities are always nontrivial set functions, and they serve for the analysis of local questions about $W^{k,p}$ functions, as well as $W^{k,p}_\infty$ functions. (cf. [8]).

(1.6) The *Cauchy transform*, $\mathcal{C}$, is defined for test functions by

$$(\mathcal{C}\phi)(\omega) = \frac{1}{\pi}\int_{\mathbf{C}} \frac{\phi(z)}{z-\omega} dxdy.$$

It is extended to more general distributions f by

$$\langle \phi, \mathcal{C}f\rangle = -\langle \mathcal{C}\phi, f\rangle,$$

whenever $\phi \in C^\infty_{\mathrm{cs}}$. For instance, this defines $\mathcal{C}f$ whenever f has compact support, or $f \in L^p$ for some $p > 2$.

The Cauchy transform inverts the $\overline{\partial}$ operator, i.e.

$$\bar\partial \mathcal{C} f = f,$$

whenever f is a distribution having compact support or belongs to some L^p.
The *Beurling transform*, $\mathcal{B}$, is defined for test functions by

$$(\mathcal{B}\phi)(\omega) = \frac{-\mathrm{PV}}{\pi}\int_{\mathbf{C}} \frac{\phi(z)}{(z-\omega)^2} dxdy.$$

It is extended to more general distributions f by

$$\langle \phi, \mathcal{B}f\rangle = \langle \mathcal{B}\phi, f\rangle,$$

We have

$$\mathcal{B}f = \partial \mathcal{C} f,$$

for each distribution f for which the right–hand–side is defined.

The Beurling transform is an example of a Calderon–Zygmund singular integral operator, so [10, Chapter 1] there exists a universal constant $A > 0$ such that if $f \in L^p$, then

$$\|\mathcal{B}f\|_p \leq \begin{cases} \dfrac{A}{p-1}\|f\|_p, & 1 < p < 2, \\ Ap\|f\|_p, & 2 \leq p < +\infty. \end{cases}$$

Each extended–real–valued Lebesgue–measurable function $f : \mathbf{C} \to [-\infty, +\infty]$ has an approximate limit, ap–$\lim_{z\to a} f(z)$, at area–almost every point $a \in \mathbf{C}$[F]. We say that an extended real-valued function $f : \mathbf{C} \to [-\infty, +\infty]$ is a *precise function* if $f(a) =$ ap–$\liminf_{z\to a} f(z)$ for each $a \in \mathbf{C}$ (— recall that the ap–lim–$\inf_{z\to a} f(z)$ is the supremum of the set of all those real numbers t such that

$$\frac{|\{z \in \mathbf{B}(a, r) : f(z) < t\}|}{\pi r^2} \to 0, \text{ as } r \downarrow 0.)$$

We say that a complex–valued function is precise if its real and imaginary parts are precise. Thus the equivalence class of each extended real-valued measurable function, with respect to the relation of a.e. equality, has a *unique* precise element. Also, the equivalence class of every locally–bounded complex–valued measurable function has a unique precise element.

We will have occasion to use a topology which we call the *p–fine topology* associated to a number $p \in [1, 2]$. For our present purposes, we may conveniently describe this topology as the least topology $\mathcal{T}$ such that each precise locally–bounded *real-valued* $W^{1,p}_{\mathrm{loc}}$ function is continuous, as a map of $\mathbf{C}$ with the topology $\mathcal{T}$ into $\mathbf{R}$, with the usual topology (induced by the Euclidean metric). Terms such as *p–fine–int(erior)*, *p*–fine neighbourhood, and so on, should be understood in terms of the *p*–fine topology. Unqualified topological terms should be understood in terms of the usual topology.

2. Statement of Results.

The analytic capacity $\bar\partial$–F_∞–cap associated to an F_∞ space characterises the removable singularities for analytic functions of that space (cf. [9]). In the present situation, this means that a compact set $E \subset \mathbf{C}$ is removable for all analytic functions belonging to $W^{k,p}_\infty$ (or $W^{k,p}$) if and only if $\gamma_{k,p}(E) = 0$.

The space $W^{1,\infty}$ evidently coincides with $W^{1,\infty}_\infty$. As is well–known, it also coincides with Lip1 (Rademacher's Theorem, cf. [2]). Nguyen [7] showed that the compact sets of removable singularities for Lip1 analytic functions are precisely the sets of area zero, and Hruschev (cf. [4]) later gave quite a simple duality argument to show that the corresponding analytic capacity is actually comparable to (i.e. within constant multiplicative bounds of) area. Thus there exists a constant $\kappa > 0$ such that

$$\gamma_\infty(E) \le |E| \le \kappa\gamma_\infty(E),$$

whenever $E \subset \mathbf{C}$ is compact.

We will not re–prove these results, but we include them where appropriate in the statements below about general $W^{k,p}_\infty$.

Our first result identifies the null–sets of all but one $\gamma_{k,p}$, in real–variable terms.

Theorem 1. *(i) For* $1 \le p \le \infty$,

$$\gamma_{1,p}(E) = 0 \iff |E| = 0.$$

(ii) For (a) $k = 2$ *and* $2 < p \le +\infty$, *or (b)* $k > 2$ *and* $1 \le p \le +\infty$,

$$\gamma_{k,p}(E) = 0 \iff \mathit{int}\,(E) = \emptyset.$$

(iii) For $1 < p \leq 2$,

$$\gamma_{2,p}(E) = 0 \iff p - \textit{fine–int}\,(E) = \emptyset.$$

We offer no real–variable classification of the null–sets of $\gamma_{2,1}$. This problem remains open. One direction of (iii) works: If 1–fine–int(E) is empty, then $\gamma_{2,1}(E) = 0$.

Next, we consider quantitative estimation of $\gamma_p = \gamma_{1,p}$, beginning with the Beurling–invariant cases ($1 < p < \infty$).

Theorem 2. *There exists* $A > 0$ *such that for all* E *compact in* $\mathbf{C}$, *with diam*$E \leq d$, *we have*
(i) for $1 < p < 2$,

$$\frac{p-1}{A}|E|^{\frac{1}{q}} \leq \gamma_p(E) \leq A(1+d^2)|E|^{\frac{1}{q}}.$$

and
(ii) for $2 \leq p < \infty$,

$$\frac{1}{Ap}|E|^{\frac{1}{q}} \leq \gamma_p(E) \leq A(1+d^2)|E|^{\frac{1}{q}}.$$

The Nguyen–Hruschev result covers the case $p = \infty$, so that leaves $p = 1$. The fact that the null-sets of γ_1 are the exactly the sets of area zero might suggest that perhaps the capacity is simply a function of area, at least in the small. We will see that the γ_1 capacity of a ball $\mathbf{B}(0, r))$ is comparable to $1/(\log 1/r)$ as $r \downarrow 0$, so the only possible function of area (up to bounded equivalence) is the function $\ell(E) = 1/\log(1/|E|)$.

We obtain a 'local' lower bound of this form for the capacity:

Theorem 3. *Given* $d > 0$, *there exists* $A > 0$ *such that*

$$\ell(E) \leq A \cdot \gamma_1(E),$$

whenever diam$(E) \leq d$.

However, we show by example that there is no corresponding local upper bound: for each $d > 0$ and $A > 0$, there is a set E of diameter less than d whose γ_1 capacity is greater than $A \cdot \ell(E)$.

Thus ℓ is not even locally equivalent to γ_1. It follows that γ_1 is not locally–equivalent to a function of area. So the problem of giving a satisfactory local real–variable description of this analytic capacity remains open.

We prove Theorem 1 in section 3, and Theorem 2 in section 4. In section 5, we estimate the $\gamma_{1,1}$ capacity of balls. In section 6, we prove Theorem 3, and in section 7 we give an example to show that the corresponding upper bound is false in general.

In the final section, we provide some rough bounds for $\gamma_{k,p}$, in case $k > 1 + 2/p$, i.e. in case $W^{k,p}$ consists of C^1 functions.

Theorem 4. *Let* $1 \leq p \leq +\infty$. *Suppose* $k > 1 + 2/p$. *Let* q *be the conjugate index to* p. *Let* $E \in \mathbf{C}$ *be compact, and* $d(z) = \mathit{dist}(z, \mathbf{C} \sim E)$. *Then*

$$\kappa \|d^{k-1}\|_{L^q} \leq \gamma_{k,p}(E) \leq \lambda_{k,p}\|d^{k-1-2/p}\|_{L^1},$$

where the constant κ *depends on* p, *but not on* E.

3. Proof of Theorem 1.

Proof of (i): Let $E \subset \mathbf{C}$ be a compact set, $|E| = 0$, and let f be admissable for E. We have to show that $f'(\infty) = 0$. But this is immediate from the formula

$$f'(\infty) = \frac{1}{\pi} \int\limits_E \bar{\partial} f dxdy. \tag{1}$$

For the converse, suppose that E has positive area. Then by Nguyen's Theorem, there exists $f \in W^{1,\infty}$ such that f is holomorphic off E and $f'(\infty) \neq 0$. Then, by Hölder's inequality, f belongs to each $W^{1,p}_\infty$, for $p \in [1, +\infty]$. So $\gamma_p(E) > 0$, as required. QED

Proof of (ii): Let $k > 2$ or $k = 2, p > 2$. Then each of the spaces $W^{k,p}_\infty$ is a subset of C^1, the space of continuously–differentiable functions. Thus it is evident that compact sets with empty interior are removable singularities for $W^{k,p}_\infty$ analytic functions. Conversely, if a compact set E has nonempty interior, then its interior supports a nonzero nonnegative test function, say ϕ. Then $\mathcal{C}\phi$ is a C^∞ function, analytic off E, has nonzero derivative at ∞, and belongs to each $W^{k,p}_\infty$. Thus $\gamma_{k,p}(E) > 0$. QED

Proof of (iii): In proving this part, we shall use Adam's result that $\max\{f, g\}$ and $\min\{f, g\}$ belong to $W^{1,p}$ whenever f and g do [3, Chapter 1].

Fix p, with $1 < p \leq 2$.

Suppose that the p–fine–interior of E is empty. Let f be $\bar{\partial}$–$W^{2,p}_\infty$–admissible for E. Then the function $\bar{\partial} f$ belongs to $W^{1,p}_\infty$, and is supported on E. Let g be the real part of $\bar{\partial} f$. Fix $n \in \mathbf{N}$. Let g_n be the precise representative of $\max\{-n, \min\{n, g\}\}$. Then g_n is real–valued and belongs to $W^{1,p}$, $g_n = 0$ off E, so $g_n = 0$ on E as well. Thus g takes only the values 0, $+\infty$, and $-\infty$ area–almost–everywhere. Since g is integrable, $g = 0$ a.e.. Similarly, the imaginary part of $\bar{\partial} f$ vanishes a.e.. Hence $\bar{\partial} f = 0$ area–almost–everywhere. By Weyl's Lemma, this means that f is entire. Applying formula (1) again, we have $f'(\infty) = 0$. Thus $\gamma_{2,p}(E) = 0$.

For the converse, suppose that p–fine–int(E) is nonempty. Then (using truncation) we may choose a single precise bounded real–valued function $h \in W^{1,p}$ such that $h > 1$ at some point of p-fine-int(E) (and hence $h > 1$ on some subset of p-fine-int(E) having positive area), and $h = 0$ off E.

Let $g = \max\{h, 0\}$. Then $g \in W^{1,p}$, $g \geq 0$, $g = 0$ off E, and $g > 0$ on a set of positive area.

Let $f = \mathcal{C}g$. Then $f \in W^{2,p}_\infty$ (by the Calderon–Zygmund theory), f is analytic off E, and

$$f'(\infty) = \int_{\mathbf{C}} g dxdy > 0.$$

Thus $\gamma_{2,p}(E) > 0$.

This concludes the proof. QED

We remark that the first part of the proof of part (iii) works also for $p = 1$, so the compact sets with empty 1–fine–interior are $\gamma_{2,1}$–null. The converse is probably false. A real–variable characterisation of the null sets of this capacity is lacking.

4. Proof of Theorem 2.

Fix E compact in $\mathbf{C}$. Let $f = \mathcal{C}(\chi_E)$, where χ_E is the characteristic function of E.

From the definition it is easily seen that $f'(\infty) = |E|/\pi$.

To get the lower bound for γ_p, we proceed to obtain an upper estimate for the $W^{1,p}_\infty$ norm of f, by estimating

$$\|\partial f\|_{L^p_\infty} + \|\overline{\partial} f\|_{L^p_\infty}.$$

First,

$$\|\overline{\partial} f\|_{L^p_\infty} = \|\chi_E\|_{L^p_\infty} = |E|^{\frac{1}{p}}.$$

Next,

$$\begin{aligned}\|\partial f\|_{L^p_\infty} = \|\mathcal{B}(\chi_E)\|_{L^p_\infty} &\le \begin{cases} \dfrac{A}{p-1}\|\chi_E\|_{L^p_\infty}, & 1 < p < 2; \\ Ap\|\chi_E\|_{L^p_\infty}, & 2 \le p < +\infty, \end{cases} \\ &\le \begin{cases} \dfrac{A}{p-1}|E|^{\frac{1}{p}}, & 1 < p < 2; \\ Ap|E|^{\frac{1}{p}}, & 2 \le p < +\infty. \end{cases}\end{aligned}$$

Thus

$$\|f\|_{W^{1,p}_\infty} \le \begin{cases} \dfrac{A}{p-1}|E|^{\frac{1}{p}}, & \text{if } 1 < p < 2; \\ Ap|E|^{\frac{1}{p}}, & \text{otherwise.} \end{cases}$$

Let $g = f/\|f\|_{W^{1,p}_\infty}$. Then $\|g\|_{W^{1,p}_\infty} = 1$, and thus

$$\gamma_p(E) \ge g'(\infty) \ge \begin{cases} \dfrac{p-1}{A\pi}|E|^{\frac{1}{q}}, & \text{if } 1 < p < 2; \\ \dfrac{1}{Ap\pi}|E|^{\frac{1}{q}}, & \text{otherwise.} \end{cases}$$

Next, we prove the upper bound.

Let f be $\overline{\partial} - W^{1,p}_\infty$–admissable for E. Let $g = \overline{\partial} f$. Then g belongs to the unit ball of L^p_∞ and $g = 0$ off E.

We may cover E using $N = 100(1+d^2)$ balls $B_1,\ldots,B_N$, of diameter 1. Then, using formula (1) of section 3 and Hölder's inequality, we get

$$\begin{aligned}|f'(\infty)| &\le \frac{1}{\pi}\int_E |g|dxdy \\ &\le \frac{1}{\pi}\sum_{j=1}^N \int_{E\cap B_j} |g|dxdy \\ &\le A(1+d^2)|E|^{1/q},\end{aligned}$$

Therefore, $\gamma_p(E) \le A(1+d^2)|E|^{1/q}$. This concludes the proof of Theorem 2. QED

5. The $W^{1,1}_\infty$ Capacity of Balls.

We consider the γ_1 capacity of the ball $B = \mathbf{B}(0,R)$. We assume that R is small, say $R < 1/2$.

Let f be admissable for B. Define g to be the circular symmetrisation of f:

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta} f(e^{i\theta}z)d\theta.$$

Then one readily checks that

$$g(z) = \frac{f'(\infty)}{z} \text{ off } E, \qquad \|g\|_{W^{1,1}_\infty} \le \|f\|_{W^{1,1}_\infty} \le 1, \qquad \text{and } |g'(\infty)| = |f'(\infty)|.$$

Thus g is also admissible, has the same derivative at ∞, and has the special form const$/z$, off E. We deduce that the capacity is the same as

$$\sup\left\{|a| : a > 0, a/z \text{ has an extension across } E \text{ that lies in the unit ball of } W^{1,1}_\infty\right\}.$$

Thus $\gamma_1(B)$ is exactly the reciprocal of

$$\inf\left\{\|f\|_{W^{1,1}_\infty} : f = \frac{1}{z} \text{ off } E\right\}.$$

Consider, in particular, the function

$$h(z) = \begin{cases} \dfrac{\overline{z}}{R^2}, & \text{if } |z| < R; \\ \dfrac{1}{z}, & \text{otherwise.} \end{cases}$$

We have

$$\partial h = \begin{cases} 0, & |z| < R, \\ -1/z^2, & |z| > R, \end{cases}$$

$$\bar\partial h = \begin{cases} 1/R^2, & |z| < R, \\ 0, & |z| > R, \end{cases}$$

Let D be any disk of radius 1. Then a routine estimate gives

$$\int_D |\partial h| dxdy \le A \log\frac{1}{R}.$$

$$\int_D |\bar\partial h| dxdy \le A.$$

Thus

$$\|h\|_{W^{1,1}_\infty} \le A\log(1/R).$$

On the other hand, if f is any element of $W^{1,1}_\infty$ having $f(z)=1/z$ off B, then

$$\|f\|_{W^{1,1}_\infty} \geq \int_{\mathbf{B}(0,1)\sim B} \frac{rdrd\theta}{r^2} \geq A^{-1}\log(1/R).$$

Combining these facts, we see that

$$\frac{A^{-1}}{\log(1/R)} \leq \gamma_1(\mathbf{B}(0,R)) \leq \frac{A}{\log(1/R)}$$

for $0<R<1/2$.

6. Proof of Theorem3: a lower bound for $\gamma_1(E)$.

It suffices to prove the case $d=1$.

Consider a set E contained in $B(0,1)$.

Fix p, with $1<p<2$.

By Theorem 2, we may choose a function $f\in W^{1,p}_\infty$ (— in fact we could use a multiple of the function $\mathcal{C}\chi_E$ —) such that f is $\bar\partial$–$W^{1,p}_\infty$–admissible for E, and

$$f'(\infty) \geq A(p-1)|E|^{1/q},$$

where q is the conjugate index to p.

But Hölder's inequality shows that, for any $g\in L^p_{\mathrm{loc}}$ and any ball B of radius 1, we have

$$\int_B |g|dxdy \leq \left(\int_B |g|^p dxdy\right)^{1/p}\cdot \pi^{1/q}.$$

This implies that

$$\|g\|_{L^1_\infty} \leq \|g\|_{L^p_\infty}\cdot \pi^{1/q}.$$

Applying this to $g=|\nabla f|$, we see that $\pi^{-1/q}f$ is $\bar\partial$–$W^{1,1}$–admissible. Therefore,

$$\gamma_1(E) \geq A(p-1)(|E|/\pi)^{1/q}.$$

For each $\alpha>0$, we have (by calculus)

$$\begin{aligned}\sup_{1<p<2}(p-1)\alpha^{1/q} &\leq 2\cdot \sup_{0<x<1/2} x\alpha^x\\ &= 2\cdot\max\left\{\frac{1}{e\log 1/\alpha},\alpha^{1/2}\right\}.\end{aligned}$$

Applying this with $\alpha=|E|/\pi$, and bearing in mind that $E\subset\mathbf{B}(0,1)$, we obtain

$$\gamma_1(E)\geq A\ell(E),$$

as required. QED

7. $\gamma_1 \neq \mathbf{O}(\ell)$**.**
Let $d > 0$ and $\kappa > 0$. We now construct a set E, with diameter less than d, such that $\gamma_1(E) \geq \kappa \cdot \ell(E)$.

The Beurling transform of the characteristic function χ_r of the ball $B(0,r)$ is given by

$$\mathcal{B}\chi_r(z) = \begin{cases} 0, \text{ if } |z| < r; \\ \dfrac{-r^2}{z^2}, \text{ otherwise.} \end{cases}$$

Let $0 < s < r + s < d$ and let E denote the annulus

$$E = \{z \in \mathbf{C} : r < |z| < r + s\}.$$

Then $\chi_E = \chi_{r+s} - \chi_r$, so we get

$$\mathcal{B}\chi_E(z) = \begin{cases} 0, \text{ if } |z| < r, \\ \dfrac{r^2}{z^2}, \text{ if } r < |z| < r+s, \\ \dfrac{-s(2r+s)}{z^2}, \text{ if } |z| > r+s. \end{cases}$$

If s/r and r are sufficiently small, we conclude that

$$\|\mathcal{B}\chi_E\|_{L^1(\mathbf{B}(0,1))} \leq 4\pi rs \log \frac{1}{r}.$$

Let $h = \chi_E/|E|$. Then $\operatorname{spt} h \subset E$, $\int h dx dy = 1$, and

$$\|h\|_{L^1} = 1 \leq \frac{1}{2\pi\kappa} \log \frac{1}{|E|},$$

provided $|E|$ is small enough. Also,

$$\|\mathcal{B}h\|_{L^1(\mathbf{B}(0,1))} \leq 8 \log \frac{1}{r} \leq \frac{1}{2\pi\kappa} \log \frac{1}{|E|},$$

provided s is sufficiently smaller than r. Now let

$$g = \frac{\kappa\pi}{\log(1/|E|)} \mathcal{C}h.$$

Then g is $\overline{\partial} - W^{1,1}_\infty$−admissible for E, and so

$$\gamma_1(E) \geq |g'(\infty)| = \left| \frac{\kappa\pi}{\pi \log 1/|E|} \int h dx dy \right| > \kappa\ell(E),$$

as required.

8. The Cases when $W^{k,p} \subset C^1$.

For this section, we suppose that we are in the situation of case 2° of Theorem 1, i.e. $k \geq 3$ (and p is unrestricted), or else $k = 2$ and $p > 2$. In these cases, the elements of $W^{k,p}$ are C^1 functions, and the null–sets of $\gamma_{k,p}$ are precisely the sets with empty interior.

Suppose that $\mathrm{int}(E)$ is nonempty. We wish to give a quantitative estimate for $\gamma_{k,p}(E)$.

We fix E compact, of diameter less than 1, and abbreviate $d(z) = \mathrm{dist}(z, \mathbf{C} \sim E)$.

Lemma 1. *Let $\phi : [0, +\infty) \to [0, +\infty)$ be any continuous function such that for some $\kappa > 0$, we have*

$$\frac{\phi(r)}{\kappa} \leq \phi(2r) \leq \kappa\phi(r).$$

Then there exist constants $\mu > 0$ (depending on κ, but independent of E) and $c_k > 0$ (independent of κ and of E), and a C^∞ function $\delta : \mathbf{C} \to [0, +\infty)$, such that

$$\frac{\phi(d(z))}{\mu} \leq \delta(z) \leq \mu\phi(d(z)),$$

whenever $z \in \mathbf{C}$, and

$$|\nabla^k \delta(z)| \leq \mu \cdot c_k \cdot \frac{\phi(d(z))}{d(z)^k},$$

whenever $k \in \mathbf{N}$ and $z \in \mathbf{C}$.

Proof. The familiar Whitney covering construction (cf. [10]) provides a partition of unity $\{\psi_n\}$ on $\mathrm{int}E$ such that $0 \leq \psi_n \leq 1$, at most 100 of the $\psi_n(z)$ are nonzero at any given z, $d(z) \leq 4d(w)$ whenever $\psi_n(z)\psi_n(w) \neq 0$ for some, n, and $d(z)^k|\nabla^k\psi(z)|$ is bounded by some constant c_k, for each $k \geq 0$.

Choose $z_n \in \mathrm{spt}\psi_n$, for each n, and let

$$\delta(z) = \sum_n \phi(d(z_n))\psi_n(z), \quad \forall z \in \mathbf{C}.$$

Then one readily verifies that δ has the desired properties. QED

Proof of Theorem 4. Let q be the conjugate index to p, as usual. If we apply the lemma to $\phi(r) = r^{(k-1)q}$, then the resulting function δ has

$$\delta(z) \geq \frac{d(z)^{(k-1)q}}{\mu}, \text{ and}$$

$$\begin{aligned} \|\nabla^{(k-1)}\delta\|_{L^p_\infty} &\leq A \cdot \left(\int_{\mathbf{C}} d(z)^{(k-1)(q-1)p} dxdy \right)^{\frac{1}{p}} \\ &= A \cdot \left(\int_{\mathbf{C}} d(z)^{(k-1)q} dxdy \right)^{\frac{1}{p}}, \end{aligned}$$

so the function $k(z) = \mathcal{C}\delta(z)/(A \cdot \int d^{(k-1)q})^{1/p}$ is $\bar\partial$–$W^{k,p}_\infty$–admissible. Thus

$$\begin{aligned}\gamma_{k,p}(E) &\geq \frac{1}{\pi}\int \bar\partial k dx dy \\ &\geq \frac{\kappa \int d^{(k-1)q}}{(\int d^{(k-1)q})^{1/p}} \\ &= \kappa \|d^{k-1}\|_{L^q}.\end{aligned}$$

This establishes the desired lower bound. The upper bound is readily obtained by using the Sobolev inclusion

$$W^{k-1,p} \hookrightarrow \mathrm{Lip}(k-1-2/p)$$

and the formula (1) of section 3. QED

Problems: We list here the most interesting open problems thrown up by this investigation.

(1) Find a real–variable description of $\gamma_{1,1}$ (if possible).

(2) Identify the null–sets of $\gamma_{2,1}$ in real–variable terms (if possible).

(3) Find sharp quantitative estimates for $\gamma_{k,p}$ when $k > 2/p$.

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