REMOVABLE SINGULARITES FOR ANALYTIC FUNCTIONS OF ZYGMUND CLASS

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[Received 19 June 1990. Read Published]

ABSTRACT

It follows from a theorem of Dolzhenko, that if a compact set in the plane has zero Hausdorff measure with respect to the measure function $h(r) = r^2 \log(1/r)$, then it is removable for analytic functions of the Zygmund class. In this paper it is shown that there is a compact set N in the plane such that N has zero Hausdorff measure with respect to each measure function $h(r)$ that is $\mathrm{o}\big(r^2(\log(1/r))^{\frac{1}{2}}\big)$ and N is non–removable for some analytic function of the Zygmund class. Some related results of a real–variable nature are obtained.

1. Introduction

A function $f : \mathbf{R}^d \to \mathbf{C}$ belongs to the *Zygmund Class* ($f \in \mathrm{ZC}$ on $\mathbf{R}^d$) if f is continuous on $\mathbf{R}^d$ and there exists $\kappa > 0$ such that

$$|f(z+h) - 2f(z) + f(z-h)| \le \kappa h$$

whenever z, $h \in \mathbf{R}^d$. See [9].

This note is about removable singularities for analytic functions that belong to ZC on the plane $\mathbf{R}^2 = \mathbf{C}$. We say that a compact set $K \subset \mathbf{C}$ is $\bar\partial$–ZC–null if for each open set $U \subset \mathbf{C}$, each function $f \in \mathrm{ZC}$ that is analytic on $U \sim K$ is actually analytic on the whole of U.

We define $A(\mathbf{C} \sim K)$ to be the class of functions $f \in \mathrm{ZC}$ that are analytic on $\mathbf{C} \sim K$.

Theorem 1. *Let $K \subset \mathbf{C}$ be compact. Then K is $\bar{\partial}$–ZC–null if and only if $A(\mathbf{C} \sim K)$ consists only of affine functions $z \mapsto \alpha + \beta \cdot z$.*

Nguyen [5] proved that if K has positive area, then there exists a Lip1 function which is analytic off K, and not entire. Thus K is not $\bar{\partial}$–ZC–null. Thus area zero is necessary for $\bar{\partial}$–ZC–nullity. He also showed by example [1986] that zero area is not sufficient for $\bar{\partial}$–ZC–nullity.

One may ask about the relation between $\bar{\partial}$–ZC–nullity and Hausdorff contents other than area. Let us denote by M_h the Hausdorff content corresponding to the measure function $h : [0, \infty) \to [0, \infty)$, i.e. for $E \subset \mathbf{C}$, $M_h(E)$ denotes the infimum of sums

$$\sum_{n=1}^{\infty} h(\mathrm{diam} B_n)$$

taken over all countable coverings $\{B_n\}$ of E by open (or by closed) balls. In 1964 Dolzhenko [1] showed the following.

Dolzhenko's Theorem. *Let $f : \mathbf{C} \to \mathbf{C}$ have modulus of continuity $\omega_f(\delta)$ and be analytic off a compact set K having $M_h(K) = 0$ for the measure function $h(r) = r\omega_f(r)$. Then f is analytic on $\mathbf{C}$.*

Since we know [9] that each function $f \in \mathrm{ZC}$ has

$$\omega_f(\delta) \leq \kappa_f \delta \log \frac{1}{\delta},$$

we deduce that the condition $M_h(K) = 0$, with $h(r) = r^2 \log \frac{1}{r}$ is sufficient for $\bar{\partial}$–ZC–nullity of K. Since $M_h(K) = 0$ with $h(r) = r^2$ is not sufficient for nullity, we may ask where, between r^2 and $r^2 \log(1/r)$, the break occurs. We will show the following.

Theorem 2. *Nguyen's example N has the following properties:*

(1) N is not $\bar{\partial}$–ZC–null;

(2) $M_h(N) = 0$ *whenever*

$$h(r) = \mathrm{o}\left(r^2 \left(\log \frac{1}{r}\right)^{\frac{1}{2}}\right).$$

Thus the question is now: where, between $h(r) = r^2 \log(1/r)$ and

$$h(r) = \mathrm{o}\left(r^2 \left(\log(1/r)\right)^{\frac{1}{2}}\right)$$

does the condition $M_h(K) = 0$ cease to be sufficient for $\bar{\partial}$–ZC–nullity of K?

As a by–product we observe a few results about functions that satisfy the Zygmund condition on the real line. First, a removable singularities result for the operator $\frac{d}{dx}$ in place of $\bar{\partial}$. Solutions of $\frac{df}{dx} = 0$ are locally constant, so the result takes the following form.

Theorem 3. *There is a compact set* $K \subset \mathbf{R}$ *and a function* $f : \mathbf{R} \to \mathbf{R}$, *such that*

(1) there exists $\kappa > 0$ *with* $|f(x+h) - 2f(x) + f(x-h)| \le \kappa h$ *whenever* $x, h \in \mathbf{R}$;

(2) f *is constant on each interval contained in* $\mathbf{R} \sim K$;

(3) f *is not constant;*

(4) $M_h(K) = 0$ *whenever* $h(r) = \mathrm{o}\left(r \left(\log(1/r)\right)^{\frac{1}{2}}\right)$.

The set K we use is, in fact, an example constructed by Kahane [1969]. Indeed, it should be remarked that Nguyen's construction is essentially a higher–dimensional extension of Kahane's idea.

In the other direction, we observe the analogue of Dolzhenko's Theorem for one dimension:

Theorem 4. *Let* $f : \mathbf{R} \to \mathbf{C}$ *be locally–constant off a compact set* $K \subset \mathbf{C}$ *having* $M_h(K) = 0$ *for the measure function* $h(r) = \omega_f(r)$. *Then* f *is constant on* $\mathbf{R}$.

Combining this with the example of Theorem 3, we obtain the corollary that there exists a function $f \in \mathrm{ZC}$ on $\mathbf{R}$ whose modulus of continuity has

$$\omega_f(\delta) \neq \mathrm{o}\left(\delta \left(\log \frac{1}{\delta}\right)^{\frac{1}{2}}\right).$$

We note that Shapiro has proved that there is a Zygmund smooth function f on $\mathbf{R}$ such that

$$|f(x+h)-2f(x)+f(x+h)| = \mathrm{O}\left(|h|\left(\log(1/|h|)^{-\frac{1}{2}}\right)\right)$$

and f is locally–constant off a set of Lebesgue measure zero. This is somewhat similar–looking to Theorem 2, but is logically independent of it. There are also other constructions of singular ZC functions in one dimension, due to Piranian [7] and (indirectly) to Keldysh and Lavrentiev (cf. [2]).

In Section 2 we review the constructions of Kahane and Nguyen, and in Section 3 we prove the results.

2. The constructions of Kahane and Nguyen

(2.1) Kahane's construction is designed to produce a nonconstant, monotonic, Zygmund class function on [0,1] whose derivative exists and vanishes on an open set of measure 1.

The construction is as follows.

Denote by w_0 the line segment [0,1] and by w_j^p the intervals of the form $[p4^{-j},(p+1)4^{-j}]$ contained in w_0. He constructs a sequence of measures μ_j and their supports K_j.

Let μ_0 be the Lebesgue measure on w_0.

The measure μ_j is to be proportional to the Lebesgue measure on each w_j^p. Let $D_j(w_j^p)$ denote the density of μ_j on w_j^p, and K_j its support, that is the union of the w_j^p's with $D_j(w_j^p) \neq 0$.

To obtain μ_{j+1} from μ_j he divides each $w = w_j^p$, contained in K_j, into four equal subintervals, w^1, w^2, w^3, w^4 (these are w_{j+1}^q's) and puts

$$\begin{aligned} D_{j+1}(w^1) &= D_{j+1}(w^4) = D_j(w) - 1 \\ D_{j+1}(w^2) &= D_{j+1}(w^3) = D_j(w) + 1. \end{aligned} \tag{1}$$

Finally, he defines μ to be the weak–star limit $\lim_j \mu_j$ and $K = \cap_j K_j$.

Observe that K is the collection of all the points $x = \sum_{j=1}^{\infty} x_j 4^{-j}$ (where $x \in \{0,1,2,3\}$) such that

$$1 + \sum_{j=1}^{k} \epsilon(x_j) > 0 \quad (k = 1, 2, \cdots) \tag{2}$$

where $\epsilon(0) = \epsilon(3) = -1$ and $\epsilon(1) = \epsilon(2) = 1$.

Let $f(x) = \mu([0,x])$ be a primitive of μ. Kahane showed that $f \in$ ZC. We reproduce the argument for the reader's convenience:

For each interval I, denote the length by $|I|$ and put $D(I) = \frac{\mu(I)}{|I|}$. Then $D(w_j^p) = D_j(w_j^p)$ for all the intervals w_j^p, and it can easily be shown that for two intervals w_j' and w_j'' having an end point in common

$$|D(w_j') - D(w_j'')| \leq 2.$$

Given an interval I, suppose j is the smallest integer such that $4^{-j} \leq |I|$.

Denote the union of the w_j^p contained in I by S_j, the union of the w_{j+1}^p contained in $\mathrm{clos}(I \sim S_j)$ by S_{j+1}, the union of w_{j+2}^p contained in $\mathrm{clos}(I \sim (S_j \cup S_{j+1}))$ by S_{j+2}, and so on.

We remark that each S_{j+k} is the union of at most 6 w_{j+k}^p's. (figure 1).

With the above notation we have

$$\begin{aligned} \mu(I) &= \mu(S_j) + \mu(S_{j+1}) + \cdots \\ &= \sum_{w_j^p \subset S_j} |w_j^p| D(w_j^p) + \sum_{w_{j+1}^p \subset S_{j+1}} |w_{j+1}^p| D(w_{j+1}^p) + \cdots \\ |I| &= \sum_{w_j^p \subset S_j} |w_j^p| + \sum_{w_{j+1}^p \subset S_{j+1}} |w_{j+1}^p| + \cdots . \end{aligned} \tag{4}$$

Suppose w_{j-1}' and w_{j-1}'' intersect I (there exists at least one and at most two of them). In the sum in (4) we have

$$|D(w_j^p) - D(w_{j-1}')| \leq 1 \quad \text{if} \quad w_j^p \subset w_{j-1}',$$

$$|D(w_j^p) - D(w_{j-1}'| \leq 3 \quad \text{if} \quad w_j^p \subset w_{j-1}'',$$

and for all k

$$|D(w_{j+k}) - D(w'_{j-1})| \leq 3 + k.$$

Then

$$\begin{aligned}|\mu(I) - |I|D(w'_{j-1})| &\leq 3\sum |w_j| + (3+1)\sum |w_{j+1}| + \cdots \\ &\leq 3|I| + 6\sum_{k=1}^{\infty} k4^{-j-k} \leq 6|I|\end{aligned}$$

If now I and I' are two contiguous intervals of equal length, we can choose the same w'_{j-1} for both. Then

$$|\mu(I) - \mu(I')| \leq 12|I| \tag{5}$$

This implies that

$$|f(x+h) + f(x-h) - 2f(x)| \leq 12h \tag{6}$$

whenever x is real and h is greater than zero. Thus $f \in$ ZC.

Kahane showed that K has length zero by relating the construction of K to the standard one-dimensional discrete random walk. We shall elaborate upon his argument in Section 3.

(2.2) Nguyen's purpose was to construct a compact set $N \subset \mathbf{C}$ of zero area and a probability measure ν, supported by N, such that its Cauchy transform

$$\hat{\nu}(z) = \int \frac{d\nu}{\zeta - z}(\zeta)$$

belongs to the Zygmund class.

In describing his construction, we shall denote Lebesgue measure in the plane by m.

He starts with a unit square Q in the complex plane. For $n = 1, 2, 3, \cdots$ let G_n be the grid of closed octadic squares of size 8^{-n} which are contained in Q. The members of G_n will be denoted by Q_j^n where $j = 1, 2, 3, \cdots (64)^n$.

He divides the squares of each grid into two types, called red and green squares. The red squares of G_1 consist of the 28 squares that intesect the boundary bdy$Q$ and any other 4 squares chosen at random in the interior of Q. The remaining 32 squares are green (See Figure 2).

Proceeding inductively, for each $Q_j^{(n-1)} \in G_{n-1}$, he chooses 32 squares of G_n which are contained in $Q_j^{(n-1)}$, in such a way that 28 of them intersect the boundary bdy$Q_j^{(n-1)}$ and, as before, he chooses the remaining 4 squares arbitrarily in the interior of $Q_j^{(n-1)}$. These 32 squares he colours red, and all the remaining squares he colours green. He labels the red squares $R_j^{(n)}$ and the green squares $G_j^{(n)}$.

He defines a sequence $\{\phi_n\}$ of Rachemacher functions:

$$(1) \qquad \phi_n(z) = \begin{cases} 1 \text{ if } z \in \text{interior } G_j^{(n)} \text{ for some } j \\ -1 \quad \text{if } z \in \text{interior } R_j^n \text{ for some } j \\ 0 \quad \text{otherwise (on grid lines)} \end{cases} .$$

Inductively he defines a sequence of functions $\{f_n\}$ by setting

$$2(a) \qquad f_1(z) = \begin{cases} 1 + \phi_1(z) & , \quad \text{if } z \in Q \\ 0 & , \quad \text{otherwise} \end{cases}$$

and

2(b) $$f_{n+1} = \begin{cases} f_n(z) + \phi_{n+1}(z) & , \quad \text{if } f_n(z) > 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

From this construction it is clear that f_n assumes only non-negative integral values, $f_n \leq n+1$, and f_n is constant on the interior of any octadic square of size 8^{-n}.

Obviously

$$\int_{Q_j^{(n)}} \phi_{n+1} dm = 0$$

(because the integral of ϕ_{n+1} over the previous generation is the sum of positive and negative terms which cancel).

Using this we obtain

(3) $$\int f_n(z) dm = 1 + \int \phi_n(z) dm = 1, \quad n = 1, 2, 3, \cdots$$

and

(4) $$\int_{Q_j^{(n)}} f_{n+k} dm = \int_{Q_j^{(n)}} f_n dm$$

for all $k = 1, 2, 3, \cdots$.

Therefore, the sequence $\{f_n m\}$ converges to a unique probability Borel measure in the weak star topology. This limit is the desired measure ν, and its support is the compact set N.

The same random walk argument (see below) proves that $m(N) = 0$ also. Nguyen showed that the Cauchy transform $\hat{\nu}$ belongs to the Zygmund class. His argument involves a rather technical and nontrivial elaboration of Kahane's idea. See [6] for details.

3. Proof of results

PROOF OF THEOREM 1. Suppose K is $\bar\partial$–ZC–null, and $f \in A(\mathbf{C} \sim K)$. Then f is an entire function of Zygmund class, and hence for each fixed $a \in \mathbf{C}$, as is easily seen,

$$z \mapsto f(a+z) - f(a)$$

is an entire function that is $\mathrm{O}(|z| \log |z|)$ for large $|z|$. Thus it is an affine function, by Liouville's Theorem, and hence so is f. This proves the "only if" part of the statement.

To prove the other direction, the key idea is to use the Vitushkin localisation operator T_ϕ. For a test function $\phi : \mathbf{C} \to \mathbf{C}$ we define

$$T_\phi f = C(\phi \bar\partial f)$$

whenever f is a distribution, where C denotes the Cauchy transform, the convolution operator

$$C = \left(\frac{-1}{\pi z} \right) *$$

which inverts $\bar\partial$ on the compactly–supported distributions. It is not too hard to see that T_ϕ maps ZC into itself, and it is clear that

$$\bar\partial \, (T_\phi f) = \phi \bar\partial f,$$

so that $\mathrm{T}_\phi f$ is analytic wherever f is and off the support of ϕ. Also, $f - T_\phi f$ is analytic on the interior of the set $\phi^{-1}(1)$.

Now suppose that $A(\mathbf{C} \sim K)$ consists only of affine functions, and let $f \in \mathrm{ZC}$ be analytic in $U \sim K$ for some open set U. To see that f is actually analytic on U, it is enough to show that it is analytic on each open disc D such that $\mathrm{clos} D \subset U$. Given such a D, we may choose a test function ϕ such that $\mathrm{spt}\phi \subset U$ and $\phi = 1$ on a full neighbourhood of $\mathrm{clos} D$. Then, writing

$$f = T_\phi f + g,$$

we see that g is analytic on D and $T_\phi f$ belongs to $A(\mathbf{C} \sim K)$, and hence by hypothesis $T_\phi f$ is entire, hence f is analytic on D, as required. ∎

We need a couple of preliminaries before we can prove Theorems 2 and 3.

Consider the following symmetric random walk. The walk starts from the initial position 1 and at each stage steps one unit in the positive or negative direction with probability $\frac{1}{2}$ each. There is an absorbing barrier at zero.

Lemma 1 [3, chap. 14, section 5]**.** *The probability of extinction at the n^{th} step is zero if n is even, and is*

$$\frac{1}{n}\binom{n}{\frac{n+1}{2}}\frac{1}{2^n}$$

if n is odd. ∎

Lemma 2. *The probability that the process has not been absorbed before or at the n-th step is asymptotic to*

$$\sqrt{\frac{2}{n\pi}}.$$

PROOF. For n odd, the probability of extinction at the n–th step is, by Lemma 1,

$$\frac{1}{n}\frac{n!}{\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!2^n}$$

and, using Stirlings formula and the approximation $(1+1/n)^n \simeq e$, this is asymptotic to

$$\frac{1}{n}\frac{\sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}}{\sqrt{2\pi}e^{-\left(\frac{n+1}{2}\right)}\left(\frac{n+1}{2}\right)^{\frac{n+2}{2}}\sqrt{2\pi}e^{-\left(\frac{n-1}{2}\right)}\left(\frac{n-1}{2}\right)^{\frac{n}{2}}}2^n$$
$$\simeq \sqrt{\frac{2}{\pi}}\frac{1}{n^{\frac{3}{2}}}.$$

Therefore the probability of surviving past the n^k step is asymptotic to

$$\sqrt{\frac{2}{\pi}}\sum_{\substack{m>n\\ m\,\mathrm{odd}}}\frac{1}{m^{\frac{3}{2}}} \simeq \frac{1}{2\sqrt{\pi}}\sum_{r=\frac{n}{2}}^{\infty}\frac{1}{r^{3/2}} \simeq \frac{1}{2\sqrt{\pi}}\int_{\frac{n}{2}}^{\infty}\frac{dr}{r^{3/2}} = \sqrt{\frac{2}{n\pi}}$$

∎

Both Kahane's construction and Nguyen's construction provide models of this same random walk. Interpreting the constructions in this way proves Theorems 2 and 3. We start with the second.

PROOF OF THEOREM 3. Consider the $\epsilon(x_j)$ as independent random independent variables on the probability space $([0,1], B, dx)$, where B denotes the family of Borel subsets of $[0,1]$. The x_j model the steps of the random walk, so Kahane's set K is (apart from a countable subset) the set of walks which survive forever. The set of walks that survive beyond the n–th step is a union of tetradic intervals of length 4^{-n}, and by Lemma 2 it has length that is $\mathrm{O}(1/\sqrt{n})$. Thus, there are $\mathrm{O}(1/\sqrt{n})4^n$ intervals involved. Thus, if $h(r) = \mathrm{o}\left(r\sqrt{\log\frac{1}{r}}\right)$, we see that

$$M_h(K) \leq \mathrm{O}(1/\sqrt{n})4^n h(4^{-n}) = \mathrm{o}(1),$$

and hence $M_h(K) = 0$, as required. ■

PROOF OF THEOREM 2. In the same way, we consider the Rademacher functions of Nguyen's construction as modelling the steps of the random walk, where this time the probability space is Lebesgue measure on the Borel subsets of the unit square. Nguyen's set N is (apart from a sigma–rectifiable set) the set of walks that never terminate, and so an argument just like the last shows that for each n it may be covered by

$$\mathrm{O}(1/\sqrt{(n)})64^n$$

squares of side 8^{-n}. Thus, if $h(r) = \mathrm{o}(r^2\sqrt{\log\frac{1}{r}})$, we see that

$$M_h(N) \leq \mathrm{O}(1/\sqrt{n})64^n h(8^{-n}) = \mathrm{o}(1),$$

and hence $M_h(N) = 0$. ■

PROOF OF THEOREM 4. It is possible to subsume Theorem 4 and Dolzhenko's theorem into a common generalisation with a real–variable proof (Dolzhenko's proof used the Cauchy integral formula), but there is a simple direct proof of Theorem 4, so we just give it.

Suppose $f : \mathbf{R} \to \mathbf{C}$ is locally–constant off a compact set $K \subset \mathbf{R}$ having $M_h(K) = 0$ for the measure function $h(r) = \omega_f(r)$. We must show that f is constant on $\mathbf{R}$.

Fix any two points a, $b \in \mathbf{R} \sim K$. It suffices to show that $f(a) = f(b)$. Fix $\epsilon > 0$. We may cover the compact set $K \cap (a, b)$ by a finite union of open intervals $I_j = (a_j, b_j)$ $(j = 1, 2, 3, \ldots, N)$ such that

$$\sum_j h(b_j - a_j) < \epsilon.$$

We may also arrange that

$$a < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < b.$$

Then since

$$\begin{aligned} f(a) &= f(a_1), \\ f(b_j) &= f(a_{j+1}), \qquad (1 \le j \le n-1) \\ f(b_n) &= f(b), \end{aligned}$$

we obtain

$$\begin{aligned} |f(b) - f(a)| &= \left| \sum_j (f(b_j) - f(a_j)) \right| \\ &\le \sum_j \omega_f(b_j - a_j) \quad < \epsilon. \end{aligned}$$

Thus $f(a) = f(b)$, as required. ■

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Draft 6. 27/10/90